Forced convex n-gons in the plane*

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March 2, 1997

Abstract

In a seminal paper from 1935, Erdős and Szekeres showed that for each n there exists a least value $g(n)$ such that any subset of $g(n)$ points in the plane in general position must always contain the vertices of a convex n-gon. In particular, they obtained the bounds

$$2^{n-2}+1 \leq g(n) \leq \binom{2n-4}{n-2}+1 ,$$

which have stood unchanged since then. In this note we remove the $+1$ from the upper bound for $n \geq 4$.

1. The main result

In 1935, Paul Erdős and George Szekeres published a short paper "A combinatorial problem in geometry" [1] which was destined to have a profound influence on the development of combinatorics (and especially Ramsey theory) during the next 60 years (cf. [3]). In particular, in this paper Erdős and Szekeres rediscovered Ramsey's theorem, which had only just appeared (unknown to them) five years earlier. Their investigations arose from a geometrical question of the talented young mathematician Esther Klein (soon to become Mrs. Szekeres). She asked, "Is it true that for every n, there is a least value $g(n)$ such that any set X of $g(n)$ points in the plane in general position always contains the vertices of a convex n-gon?"

Erdős and Szekeres gave several proofs of the existence of $g(n)$ in [1] and established the following bounds:

$$2^{n-2}+1 \leq g(n) \leq \binom{2n-4}{n-2}+1 . \tag{1}$$

They also conjectured that the lower bound in (1) in fact always holds with equality. This is known [2] to be the case for $n \leq 5$.

Despite repeated attempts over the years, no general improvement on (1) has been found.

In this note, we make a very small improvement on the upper bound of (1). Namely, we show

$$g(n) \leq \binom{2n-4}{n-2} \tag{2}$$

for $n \geq 4$.

*To appear in *Discrete and Computational Geometry*

†Research supported in part by NSF Grant No. DMS 95-04834

While this is admittedly rather modest, we hope that it might suggest methods which could give rise to more substantial reductions in the upper bound.

Proof of (2): By an *m-cap* we mean a sequence of m points $x_1, x_2, \ldots, x_m$ such that the polygonal path connecting them is concave, i.e., the x_i have increasing x-coordinates and the path from x_1 to x_m turns clockwise at each intermediate vertex. Similarly, an *m-cup* is a set of points $y_1, y_2, \ldots, y_m$ with increasing x-coordinates such that the polygonal path joining them is convex, i.e., the path from y_1 to y_m always turns counterclockwise.

Figure 1: Caps and cups

The following result from [1] follows easily by induction.

Lemma 1. *If $X \subset \mathbb{E}^2$ is in general position and $|X| > \binom{a+b-4}{a-2}$ then X contains either an a-cap or a b-cup.*

In fact, as shown in [1], this bound is sharp.

Theorem *If $X \subset \mathbb{E}^2$ is in general position and $|X| \geq \binom{2n-4}{n-2}$ for $n \geq 4$, then X contains the vertices of a convex n-gon.*

Proof: Suppose the contrary. Rotate X if necessary so that no line determined by two points of X is either horizontal or vertical. We can further assume without loss of generality that all lines determined by two points of X have slopes less than 0.1 in absolute value (by uniformly compressing X in the y-direction, if necessary).

Define $A := \{x \in X : x$ is the left-hand endpoint of some $(n-1)$-cap in $X\}$.

Case 1. $|A| > \binom{2n-5}{n-3}$.

Then by Lemma 1, A contains an $(n-1)$-cup, say, $y_1, y_2, \ldots, y_{n-1}$. Since $y_{n-1} \in A$, there

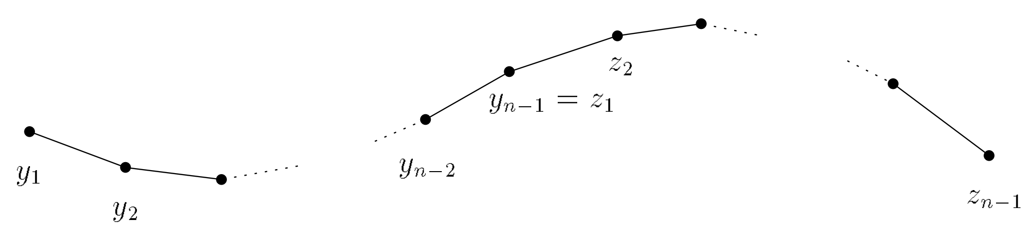

Figure 2: A cup joining a cap

exists an $(n-1)$-cap $y_{n-1} = z_1, z_2, \ldots, z_{n-1}$ in X. However, this is impossible since either y_1, $y_2, \ldots y_{n-1}, z_2$ is an n-cup, or $y_{n-2}, z_1, z_2, \ldots, z_{n-1}$ is an n-cap (see Fig. 2).

Case 2. $|A| < \binom{2n-5}{n-3}$.

Then $B := X \setminus A$ satisfies $|B| > \binom{2n-4}{n-2} - \binom{2n-5}{n-3} = \binom{2n-5}{n-3}$ and, similarly as in Case 2, we reach a contradiction.

This leaves as the only possibility:

Case 3. $|A| = |B| = \binom{2n-5}{n-3} = \frac{1}{2}\binom{2n-4}{n-2}$.

For any $b \in B$, consider the set $A \cup \{b\}$. Since this set has size greater than $\binom{2n-5}{n-3}$ then by Lemma 1, it contains an $(n-1)$-cup, say with right-hand endpoint y. Now, if $y \in A$ then as in Case 1, we reach a contradiction. Hence we must have $y = b$.

Thus, each $b \in B$ is the right-hand endpoint of an $(n-1)$-cup with left-hand endpoint in A. It follows in a similar way that each $a \in A$ is the left endpoint of an $(n-1)$-cap with right-hand endpoint in B.

We now form a directed bipartite graph G with vertex sets A and B, and edge set E consisting of all pairs (u, v), where either $u \in A$ is the left-hand endpoint and $v \in B$ is the right-hand endpoint of some $(n-1)$-cap in X, or $v \in A$ is the left-hand endpoint and $u \in B$ is the right-hand endpoint of some $(n-1)$-cup in X.

By the preceding remarks, it follows that all vertices of G have outdegree at least one. This implies G has some (directed) cycle $C = a_{i_1} b_{i_1} \cdots a_{i_r} b_{i_r}$.

Now consider an edge $(a, b) \in E$. Let $L^+(a)$ denote the half-line starting at a and going down with slope 0.1, and let $R^-(b)$ denote the half line starting at b and going down with slope -0.1. Also, let $S(a, b)$ denote the line segment joining a and b. Finally, let $Y(a, b)$ denote the region of $\mathbb{E}^2$ (strictly) below the path $L^+(a)S(a, b)R^-(b)$ (see Fig. 3).

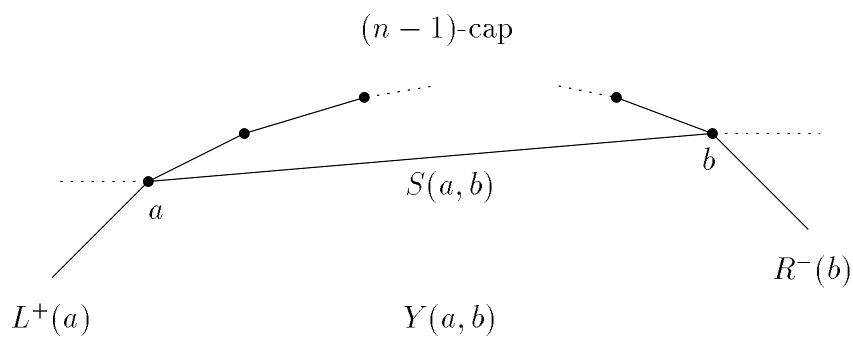

Figure 3:

Claim 1. *X has no point in $Y(a, b)$.*

Otherwise, if $x \in X \cap Y(a, b)$ then the $(n-1)$-cap spanned by (a, b) together with x forms a convex n-gon in X, which is a contradiction. ∎

By an analogous argument for $(b,a) \in E$, with $L^-(a)$, $R^+(b)$, $Y(b,a)$ defined accordingly (see Fig. 4), we also see that $Y(b,a)$ can contain no point of X.

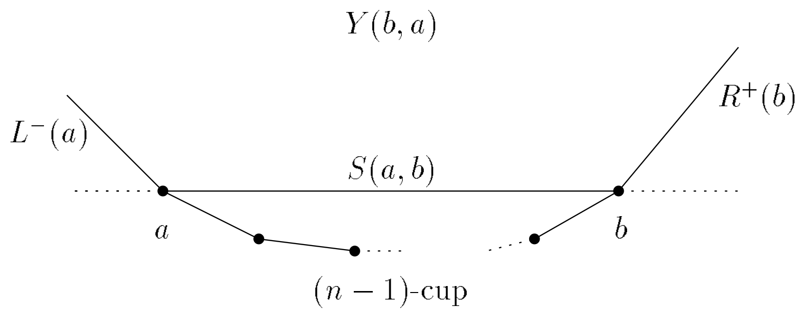

Figure 4:

Next, consider two connected edges (a,b) and (b,a') in E. We cannot have $a = a'$, since if we did, then X would contain a convex $(2n-4)$-*gon* (formed by the $(n-1)$-cap and $(n-1)$-cup spanned by a and b), which is impossible.

Claim 2. *a' must lie* above *the line through a and b.*

Proof: Suppose not. Then from the geometry of the situation (see Fig. 5), either $a' \in Y(a,b)$ or $a \in Y(b,a')$, a contradiction. A similar argument shows if $(b,a) \in E$ and $(a,b') \in E$ then b'

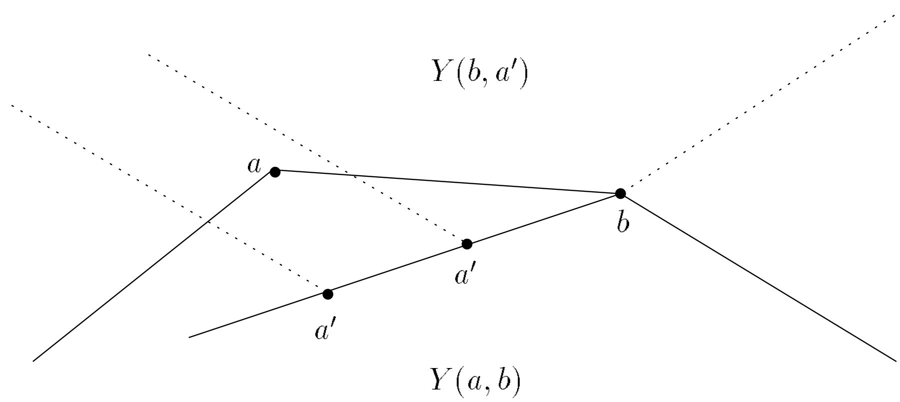

Figure 5:

must lie below the line through b and a. ∎

Finally, consider the cycle $C = a_{i_1}b_{i_1}\cdots a_{i_r}b_{i_r}$ occurring in G. If $r = 1$ then we find a convex $(2n-4)$-gon, which is impossible. So, we may assume $r \geq 2$. By Claim 2, each of the angles between adjacent edges, $a_{i_1}b_{i_1}, b_{i_1}a_{i_2}, a_{i_2}b_{i_2}\cdots a_{i_r}b_{i_r}, b_{i_r}a_{i_1}$ must turn in a counterclockwise direction. Hence, the lines through the consecutive edges $a_{i_1}b_{i_1}, b_{i_1}a_{i_2}, a_{i_2}b_{i_2}\cdots$ have increasing slopes, and any pair of these lines intersects at an angle of less than $2 \arctan 0.1 < 12°$. However, since all of the slopes of the lines are between -0.1 and 0.1, and C is a cycle, we reach a contradiction. ∎

We are inclined to believe (as did Erdős and Szekeres) that the lower bound $2^{n-2}+1$ is the true value of $g(n)$. However, we admit that there is little real evidence yet for this belief.

References

[1] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compositio Math.* **2** (1935), 463-470.

[2] P. Erdős and G. Szekeres, On some extremum problems in elementary geometry, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **3–4** (1961), 53–62.

[3] R. L. Graham and J. Nešetřil, Ramsey theory in the work of Paul Erdős, *The Mathematics of Paul Erdős*, (R. L. Graham and J. Nešetřil, eds.), Springer Verlag, Heidelberg, 1996.