

# Quasi-random hypergraphs revisited

**Corrigendum: Quasi-random classes of hypergraphs,  
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## Abstract

The quasi-random theory for graphs mainly focuses on a large equivalent class of graph properties each of which can be used as a certificate for randomness. For  $k$ -graphs (i.e.,  $k$ -uniform hypergraphs), an analogous quasi-random class contains various equivalent graph properties including the  *$k$ -discrepancy property* (bounding the number of edges in the generalized induced subgraph determined by any given  $(k-1)$ -graph on the same vertex set) as well as the  *$k$ -deviation property* (bounding the occurrences of “octahedron”, a generalization of 4-cycle). In a 1990 paper [1], a weaker notion of  $l$ -discrepancy properties for  $k$ -graphs was introduced for forming a nested chain of quasi-random classes, but the proof for showing the equivalence of  $l$ -discrepancy and  $l$ -deviation, for  $2 \leq l < k$ , contains an error. An additional parameter is needed in the definition of discrepancy, because of the rich and complex structure in hypergraphs. In this note, we introduce the notion of  $(l, s)$ -discrepancy for  $k$ -graphs and prove that the equivalence of the  $(k, s)$ -discrepancy and the  $s$ -deviation for  $1 \leq s \leq k$ . We remark that this refined notion of discrepancy seems to point to a lattice structure in relating various quasi-random classes for hypergraphs.

## 1 Introduction

The study of quasi-random graphs and hypergraphs explores the relationship among properties of graphs with special emphasis of finding equivalence classes and their classifications. For graphs, there is a large equivalence class that includes the *discrepancy property* and the *deviation property* [4]. The discrepancy property for a graph  $G$  is associated with bounding the difference between the number of edges in an induced subgraph  $S$  of  $G$  and the expected number of

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edges in  $S$  (which is basically  $|S|^2/4$  for a graph  $G$  with edge density  $1/2$ ). The discrepancy property for  $G$  is associated with bounding the difference between the number of four cycles containing an even number of edges in  $G$  and those with an odd number of edges in  $G$ . To extend the study of quasi-random graphs to  $k$ -uniform hypergraphs, (or  $k$ -graphs for short), there have been numerous attempts [1, 3, 5, 7, 8]. In the effort to extend the notion of deviation to  $k$ -graphs for  $k \geq 3$ , there is a nested sequences of  $l$ -deviation  $\text{dev}_l$ ,  $2 \leq l \leq k$  which concern the counts of so-called even “octahedra” and odd octahedra on  $2l$  vertices. To generalize the notion of discrepancy for a  $k$ -graphs  $H$  with vertex set  $V$ , one of the ways is to consider the  $l$ -discrepancy  $\text{disc}_l H$ , for a fixed  $l$ ,  $2 \leq l \leq k$ , which concerns the maximum difference of the edge counts in subgraph of  $H$  induced by any  $(l-1)$ -graph  $G$  from the expected value over all  $G$  on  $V$ . In [1, 3] it was shown that for a  $k$ -graph  $H$ , the property  $\text{disc} H = \text{disc}_k H$  and  $\text{dev} H = \text{dev}_k H$  are equivalent in the sense that for any  $\epsilon$  there exists  $\delta$  such that  $\text{disc} H \leq \delta$  implies  $\text{dev} H \leq \epsilon$ , (denoted by  $\text{disc} \Rightarrow \text{dev}$ ) and the reverse direction holds as well.

To further understand the structure for  $k$ -graphs, a natural approach is to establish a nested sequence of equivalence classes. In [1], it was shown that for  $2 \leq l \leq k$ ,  $\text{dev}_l \Rightarrow \text{disc}_l$ . However, the proof for  $\text{disc}_l \Rightarrow \text{dev}_l$  contains two cases, one of which, namely for  $2 \leq l < k$ , contains an erroneous statement. A counterexample was given in [6]. As it turns out, the hypergraphs have a richer and more intriguing structure than previously suspected (by the author). There are further extensions of the discrepancy property which we call  $(l, s)$ -discrepancy, denoted by  $\text{disc}_l^{(s)} H$ , for a  $k$ -graph  $H$  with vertex set  $V$ , where  $2 \leq l \leq k$  and  $1 \leq s \leq \binom{k}{l}$ . Roughly speaking,  $\text{disc}_l^{(s)} H$  concerns the subgraphs  $S_s$  of  $H$  which are induced by an  $l$ -graph  $G$  on  $V$  in the sense that an edge  $x$  in  $E(H)$  is in  $S_s$  if the number of  $l$ -edges in  $G$  contained in  $x$  is at least  $s$ . The previous notion of  $\text{disc}_l$  is the special case of  $\text{disc}_l^{(s)}$  with  $s = \binom{k}{l-1}$ . The paper [6] examines the case of  $\text{disc}_2^{(k)}$  which was then shown to belong to a large equivalence class of hypergraph properties including counting the appearances of a fixed “linear”  $k$ -graph  $F$  in  $H$  where “linear” means the restriction that any two edges in  $F$  intersect at at most one vertex.

With this refined notion of  $(l, s)$ -discrepancy for  $k$ -graphs, numerous questions arise. How are various known hypergraph properties related to  $\text{disc}_l^{(s)}$ ? For example, suppose we consider a generalization of linear  $k$ -graphs. We say a  $k$ -graph  $F$  is  $l$ -linear if any two edges in  $F$  intersect at no more than  $l$  vertices. We can then define the following subgraph containment property for a  $k$ -graph  $H$  on  $n$  vertices :

$P_l$ : For every  $(l-1)$ -linear  $k$ -graph  $F$  on  $r$  vertices with  $r$  vertices and  $t$  edges with  $r \geq k$ , the number  $N_F(H)$  of labelled embeddings of  $F$  in  $H$  satisfies

$$N_F(H) = (1/2)^t n^r + o(n^r).$$

It seems plausible to conjecture that  $P_l$  is equivalent to  $\text{disc}_l^{(s)}$  with  $s = \binom{k}{l-1}$

by extending the techniques in [6] for the case of  $l = 2$ . Although the above formulation is mainly for  $k$ -graph  $H$  with edge density  $1/2$ , a general definition for  $P_l$  with graphs with edge density  $p$  can be obtained in a straightforward manner by replacing  $1/2$  by  $p$ .

There are further questions just for the case of  $l = 2$ . Even for the special case of  $k = 3$  and  $s = 2$ , the discrepancy property for a 3-graph  $H$  is reduced to the following: For any subset  $S$  of vertices, the number of edges in  $H$  containing at least 2 vertices in  $S$  is about as expected. Will this property be equivalent to some modified version of the deviation property (similar to some partial “doubling” as described in [6])?

In general, for various given hypergraph properties, can they be related to the  $\text{disc}_l^{(s)}$  in some way? Do they form quasi-random equivalence classes? What are the hierarchy of these quasi-random classes? And, how effective are these properties to be used as certificates for randomness? To partially answer some of these questions, we show that  $\text{dev}_s$  is equivalent to  $\text{disc}_k^{(s)}$  for  $k$ -graphs in the remaining part of this note. Further questions and remarks concerning the lattice structure of quasi-random classes for  $k$ -graphs will be discussed in the last section.

## 2 A refined notion of the discrepancy properties for hypergraphs

We follow the notation in [3]. A  $k$ -uniform hypergraph  $H = (V, \mu_H)$  consists of a set of  $V$  of vertices of  $H$  together with a function  $\mu_H : \binom{V}{k} \rightarrow \{1, -1\}$ , called the multiplicative edge function of  $H$ . The set  $E(H) = \mu_H^{-1}(-1)$  is called the edge set of  $H$ . When there is no confusion, we call  $H$  a  $k$ -graph. For a given function  $\mu : \binom{V}{k} \rightarrow \{1, -1\}$ , denote by  $\bar{\mu}$  the extension  $\bar{\mu} : V^k \rightarrow \{1, -1\}$  by  $\bar{\mu}(v_1, \dots, v_k) = \mu(\{v_1, \dots, v_k\})$  where  $v_1, \dots, v_k$  are distinct elements of  $V$  and 1 otherwise.

**Definition.** The  $l$ -deviation of a  $k$ -graph  $H = H(E, V)$  with  $|V| = n$ , denoted by  $\text{dev}_l H$ , is defined by

$$\text{dev}_l H = \frac{1}{n^{k+l}} \sum_{\substack{v_i(0), v_i(1) \in V \\ 1 \leq i \leq l}} \sum_{\substack{w_j \in V \\ l+1 \leq j \leq k}} \prod_{\substack{\epsilon_i \in \{0,1\} \\ 1 \leq i \leq l}} \bar{\mu}_H(v_1(\epsilon_1), \dots, v_l(\epsilon_l), w_{l+1}, \dots, w_k)$$

where  $\bar{\mu}(x) = -1$  if  $x$  is an edge in  $H$  and  $\bar{\mu}(x) = 1$  otherwise.

We remark that the above definition can be generalized to focus on graphs with edge density  $p$  by defining  $\mu(x) = -p$  if  $x$  is an edge in  $H$  and  $\mu(x) = 1 - p$  otherwise.

**Definition.** For a  $k$ -graph  $H$  and a  $l$ -graph  $G$  on the same vertex set  $V$ , we define

$$\begin{aligned} E(H, G) &= \{x \in E(H) : \binom{x}{l} \subseteq E(G)\}, \\ e(H, G) &= k!|E(H, G)|. \end{aligned}$$

Namely  $e(H, G)$  counts the number of ordered subsets in  $E(H, G)$ .

**Definition.** For a  $k$ -graph  $H$  on vertex set  $V$  with  $|V| = n$ , we define  $\text{disc}_l H$  as follows:

$$\text{disc}_l H = \frac{1}{n^k} \max_G |e(H, G) - e(\bar{H}, G)|,$$

where the maximum is taken over all  $(l-1)$ -graphs  $G$  on  $V$ .

It was shown in [1, 3] that

$$\text{dev}_l H \geq (\text{disc}_l H)^{2^l}.$$

and for  $l = k$ ,

$$\text{dev}_k H \leq 4^k (\text{disc}_k H)^{2^{-k}}. \tag{1}$$

For a  $k$ -graph  $H$ , we use the notation that  $\text{dev} H = \text{dev}_k H$  and  $\text{disc} H = \text{disc}_k^{(k)} H$  which [3] mainly focused on.

It would have led to quasi-random classes for hypergraphs if a similar statement as follows holds for  $2 \leq l < k$ .

$$\text{dev}_l H \leq 4^l (\text{disc}_l H)^{2^{-l}}.$$

However, this inequality is not true for  $l \neq k$  as evidenced by the example given in [6]. So, a natural question is to find the ‘right’ equivalent discrepancy property for  $\text{dev}_l$ .

The structures in hypergraphs are indeed complex and intriguing in many ways. By considering a general family of discrepancy properties, many related questions arise.

**Definition.** For a  $k$ -graph  $H$  and an  $l$ -graph  $G$ , we define

$$\begin{aligned} E_s(H, G) &= \{x \in E(H) : \left| \binom{x}{l} \cap E(G) \right| \geq s\}, \\ e_s(H, G) &= k!|E_s(H, G)|. \end{aligned}$$

Namely  $e_s(H, G)$  counts the number of ordered subsets in  $E_l(H, G)$ . We note that for the case of  $l = k - 1$  and  $s = k$ , we have  $e(H, G) = e_k(H, G)$ .

**Definition.** For a  $k$ -graph  $H$  on  $n$  vertices, we define  $\text{disc}_l^{(s)} H$  as follows:

$$\text{disc}_l^{(s)} H = \frac{1}{n^k} \max_G |e_s(H, G) - e_s(\bar{H}, G)|,$$

where the max is taken over all  $(l-1)$ -graphs  $G$  on  $V$ .

Note that  $\text{disc} H$  is the special case  $\text{disc}_k = \text{disc}_k^{(k)}$  and  $\text{disc}_l$  is the special case  $\text{disc}_l = \text{disc}_l^{(s)}$  for  $s = \binom{k}{l-1}$ .

We remark that the above definition can be modified to focus on graphs with density  $p$  by defining  $\text{disc}_l^{(s)} H = \frac{1}{\text{vol} H} \max_G |e_s(H, G) - p \cdot e_s(\binom{V}{k}, G)|$  where  $G$  ranges over all  $(l-1)$ -graphs. Here  $\text{vol} H$  denotes the number of edges in  $H$ . For simplicity, we will mainly deal with the case of  $p = 1/2$  here.

Although we are far from fully understanding the relationship among properties  $\text{disc}_l^{(s)}$ , certain implications can be derived for the case of  $l = k$ . To simplify the notation, we write

$$\text{disc}^{(s)} H = \text{disc}_k^{(s)} H.$$

Note that for  $\text{disc}^{(s)}$ , the interesting range for  $s$  is for  $s \leq k$ .

We will prove the following two theorems to establish the equivalence implications of  $\text{dev}_s$  and  $\text{disc}^{(s)}$ .

### 3 The $s$ -deviation property implies the discrepancy property $\text{disc}^{(s)}$

**Theorem 1** For a  $k$ -graph  $H$  and  $2 \leq s \leq k$ , we have

$$\text{dev}_s H \geq (\text{disc}^{(s)} H)^{2^s}.$$

*Proof:* It suffices to show that for any given  $(k-1)$ -graph  $G$ , we have

$$\text{dev}_s H \geq (E_s(H, G) - E_s(\bar{H}, G))^{2^s}.$$

This can be proved by applying the Cauchy-Schwarz inequality on selected terms

repeatedly as follows. We consider

$$\begin{aligned}
\text{dev}_s H &= \frac{1}{n^{k+s}} \sum_{\substack{v_i(0), v_i(1) \in V \\ 1 \leq i \leq s}} \sum_{\substack{w_j \in V \\ s+1 \leq j \leq k}} \prod_{\substack{\epsilon_i \in \{0,1\} \\ 1 \leq i \leq s}} \bar{\mu}_H(v_1(\epsilon_1), \dots, v_s(\epsilon_s), w_{s+1}, \dots, w_k) \\
&= \frac{1}{n^{k+s}} \sum_{\substack{v_i(0), v_i(1) \in V \\ 1 \leq i \leq s-1}} \sum_{\substack{w_j \in V \\ s+1 \leq j \leq k}} \left( \sum_{v \in V} \prod_{\substack{\epsilon_i \in \{0,1\} \\ 1 \leq i \leq s-1}} \bar{\mu}_H(v_1(\epsilon_1), \dots, v_{s-1}(\epsilon_{s-1}), v, w_{s+1}, \dots, w_k) \right)^2 \\
&\geq \frac{1}{n^{k+s}} \sum_{\substack{w_j \in V \\ s+1 \leq j \leq k}} \sum_{\substack{v_i(0), v_i(1) \in V \\ 1 \leq i \leq s-1}}^{G,s} \left( \sum_{v \in V} \prod_{\substack{\epsilon_i \in \{0,1\} \\ 1 \leq i \leq s-1}} \bar{\mu}_H(v_1(\epsilon_1), \dots, v_{s-1}(\epsilon_{s-1}), v, w_{s+1}, \dots, w_k) \right)^2
\end{aligned}$$

where  $\sum^{G,s}$  denotes a partial sum with the restriction that the  $v_i(\epsilon_i)$  satisfy the property that  $(v_1(\epsilon_1), \dots, v_{s-1}(\epsilon_{s-1}), w_{s+1}, \dots, w_k)$  are edges in  $G$  for all  $\epsilon_i$ . Thus we have

$$\begin{aligned}
\text{dev}_s H &\geq \frac{\sum_{\substack{w_j \in V \\ s+1 \leq j \leq k}} \sum_{\substack{v_i(0), v_i(1) \in V \\ 1 \leq i \leq s-1}}^{G,s}}{n^{k+s-2}} \left( \sum_{\substack{v \in V \\ \epsilon_i \in \{0,1\} \\ 1 \leq i \leq s-1}} \bar{\mu}_H(v_1(\epsilon_1), \dots, v_{s-1}(\epsilon_{s-1}), v, w_{s+1}, \dots, w_k) \right)^2 \\
&\geq \left( \frac{1}{n^{k+s-1}} \sum_{\substack{w_j \in V \\ s \leq j \leq k}} \sum_{\substack{v_i(0), v_i(1) \in V \\ 1 \leq i \leq s-1}}^{G,s} \prod_{\substack{\epsilon_i \in \{0,1\} \\ 1 \leq i \leq s-1}} \bar{\mu}_H(v_1(\epsilon_1), \dots, v_{s-1}(\epsilon_{s-1}), w_s, w_{s+1}, \dots, w_k) \right)^2
\end{aligned}$$

We will repeat the same methods using the notation that  $\sum^{G,[j,s]}$  denotes a partial sum with the restriction that the  $v_i(\epsilon_i)$  satisfy the property that  $(v_1(\epsilon_1), \dots, v_{j-1}(\epsilon_{j-1}), w_j, \dots, w_{t-1}, w_{t+1}, \dots, w_k)$  are edges in  $G$  for all  $\epsilon_i$  and  $t \in [j, s]$ . Then we have

$$\begin{aligned}
\text{dev}_s H &\geq \left( \frac{1}{n^{k+s-1}} \sum_{\substack{w_j \in V \\ s \leq j \leq k}} \sum_{\substack{v_i(0), v_i(1) \in V \\ 1 \leq i \leq s-1}}^{G,[s,s]} \prod_{\substack{\epsilon_i \in \{0,1\} \\ 1 \leq i \leq s-1}} \bar{\mu}_H(v_1(\epsilon_1), \dots, v_{s-1}(\epsilon_{s-1}), w_s, w_{s+1}, \dots, w_k) \right)^2 \\
&\geq \left( \frac{1}{n^{k+s-2}} \sum_{\substack{w_j \in V \\ s-1 \leq j \leq k}} \sum_{\substack{v_i(0), v_i(1) \in V \\ 1 \leq i \leq s-2}}^{G,[s-1,s]} \prod_{\substack{\epsilon_i \in \{0,1\} \\ 1 \leq i \leq s-2}} \bar{\mu}_H(v_1(\epsilon_1), \dots, v_{s-1}(\epsilon_{s-2}), w_s, w_{s+1}, \dots, w_k) \right)^4 \\
&\geq \dots \\
&\geq \left( \frac{1}{n^{k+1}} \sum_{\substack{w_j \in V \\ 2 \leq j \leq k}} \sum_{\substack{v_i(0), v_i(1) \in V \\ 1 \leq i \leq 2}}^{G,[2,s]} \prod_{\epsilon_i \in \{0,1\}} \bar{\mu}_H(v_1(\epsilon_1), w_2, w_{s+1}, \dots, w_k) \right)^{2^{s-1}}
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\text{dev}_s H &\geq \left( \frac{1}{n^{k-1}} \sum_{\substack{w_j \in V \\ 2 \leq j \leq k}} \left( \frac{1}{n} \sum_{v \in V}^{G, [2, s]} \bar{\mu}_H(v, w_2, w_{s+1}, \dots, w_k) \right)^2 \right)^{2^{s-1}} \\
&\geq \left( \frac{1}{n^k} \sum_{\substack{w_j \in V \\ 1 \leq j \leq k}}^{G, [1, s]} \bar{\mu}_H(w_1, w_2, w_{s+1}, \dots, w_k) \right)^{2^s} \\
&= \left( \frac{1}{n^k} (e_s(H, G) - e_s(\bar{H}, G)) \right)^{2^s}
\end{aligned}$$

for any  $(k-1)$ -graph  $G$ . Thus we conclude that

$$\text{dev}_s H \geq (\text{disc}^{(s)} H)^{2^s}$$

□

## 4 The discrepancy property $\text{disc}^{(s)}$ implies the $s$ -deviation property

**Theorem 2** *For a  $k$ -graph  $H$  and  $2 \leq s \leq k$ , suppose that for every  $(k-1)$ -graph  $G$  on  $V$ ,*

$$|e_s(H, G) - e_s(\bar{H}, G)| \leq \epsilon n^k.$$

*Then we have*

$$\text{dev}_s H \leq 2^{k+s} \epsilon^{1/((k-s)(k-s+1)2^s)}. \quad (2)$$

*Proof:*

Assume that  $k \geq 3$  (since the case of  $k = 2$  is well understood [4]). We will first give a relative simple example for the case of  $k = 3$  and  $s = 2$  before proceeding to the general case.

Suppose that for every 2-graph  $G$  on  $V$ , we have

$$|e_2(H, G) - e_2(\bar{H}, G)| \leq \epsilon n^2.$$

We wish to show

$$\text{dev}_2 H \leq 32\epsilon^{1/8}.$$

For a vertex  $w$ , we consider the 2-graph  $H_w$  with edge set  $E(H_w) = \{y \in \binom{V}{2} : y \cup \{w\} \in E(H)\}$ . From the definition of  $\text{dev}_2 H$ , we have

$$\text{dev}_2 H = \frac{1}{n} \sum_{w \in V} \text{dev}_2 H_w$$

We consider

$$S := \{w \in V : \text{dev}_2 H_w \geq 30\epsilon^{1/8}\},$$

If  $|S| \leq 2\epsilon^{1/2}n$  then

$$\text{dev}_2 H \leq \frac{1}{n}(|S| + 30\epsilon^{1/8}n) \leq 32\epsilon^{1/8}.$$

as desired. Thus, we may assume  $|S| \geq 2\epsilon^{1/2}n$ .

For each  $w \in S$ , the fact that  $\text{dev}_2 H_w \geq \epsilon' = 30\epsilon^{1/8}$  implies, by the induction hypothesis using (1) for 2-graphs, that there exists a subset  $G_w$  (which can be viewed as a 1-graph on  $V$ ) satisfying

$$|e(H_w, G_w) - e(\bar{H}_w, G_w)| > \delta n^2$$

where  $\delta$  satisfies  $\delta \geq 16^{-4}\epsilon'^4 \geq 3\epsilon^{1/2}$ . Thus, there is a subset  $S'$  of  $S$  with  $|S'| = \epsilon^{1/2}n$  so that either

- (a)  $e(H_w, G_w) \geq \frac{1}{2}e(\binom{V}{2}, G_w) + 3\epsilon^{1/2}n^2/2$  for all  $w \in S'$ ; or
- (b)  $e(H_w, G_w) \leq \frac{1}{2}e(\binom{V}{2}, G_w) - 3\epsilon^{1/2}n^2/2$  for all  $w \in S'$ .

We will treat case (a) and omit the similar treatment for case (b).

We proceed to define the following 2-graph  $G$  on  $V$ .

$$E(G) = \{w \cup y : y \in E(G_w)\} \setminus \binom{V \setminus S'}{2}.$$

For each  $x \in E_2(H, G)$ , there are three possibilities:

- (i)  $x$  has at least two vertices in  $S'$ . There are at most  $\epsilon n^3$  such edges in  $E_2(H, G)$ .
- (ii)  $x$  has no vertex in  $S'$ . In this case,  $x$  can not contain a pair of vertices in  $G$ , contradicting to  $x \in E_2(H, G)$ .
- (iii)  $x$  has exactly one vertex  $w$  in  $S'$ . Say,  $x = \{v, u, w\}$  and  $u, v \in H_w$ . Therefore, we have

$$\begin{aligned} E_2(H, G) - E_2(\bar{H}_G) &\geq \sum_{w \in S'} (E(H_w, G_w) - E(\bar{H}_w, G_w)) - \epsilon n^3 \\ &\geq |S'| 3\epsilon^{1/2}n^2/2 - \epsilon n^3 \\ &\geq 2\epsilon n^2 \end{aligned}$$

which is a contradiction. Thus we have proved (2) for the case of  $k = 3$ .

The proof for the general  $k$  is quite similar. For a  $k$ -graph  $H$ , suppose that for every  $(k - 1)$ -graph  $G$  on  $V$ , we have

$$|e_s(H, G) - e_s(\bar{H}, G)| \leq \epsilon n^k.$$

We wish to show

$$\text{dev}_s H \leq 2^{k+s} \epsilon^{1/((k-s)(k-s+1)2^s)}.$$

For a fixed string of  $k - s$  vertices, say,  $w = (w_1, w_2, \dots, w_{k-s})$ , we consider edges in  $E(H)$  containing  $w_i$  for  $1 \leq i \leq j$ . We consider the  $(k - i)$ -graph  $H_{(w_1, \dots, w_i)}$  with edge set  $E(H_{(w_1, \dots, w_i)}) = \{y \in \binom{V}{s} : y \cup \{w_1, \dots, w_i\} \in E(H)\}$ . From the definition of  $\text{dev}_s H$ , we have

$$\begin{aligned} \text{dev}_s H &= \frac{1}{n} \sum_{w_1} \text{dev}_s H_{(w_1)} \\ &= \frac{1}{n^2} \sum_{w_1, w_2} \text{dev}_s H_{(w_1, w_2)} \\ &= \dots \\ &= \frac{1}{n^{k-s}} \sum_{w=(w_{s+1}, \dots, w_k)} \text{dev}_s H_w \end{aligned}$$

For  $w_1 \in V$ , we consider

$$S_1 := \{w_1 \in V : \sum_{w_1} \text{dev}_s H_{w_1} \geq (2^{k+s} - 2) \epsilon^{1/((k-s)(k-s+1)2^s)}\},$$

If  $|S| \leq 2\epsilon^{1/(k-s+1)}n$  then

$$\text{dev}_s H \leq \frac{1}{n} (|S| + (2^{k+s} - 2) \epsilon^{1/((k-s)(k-s+1)2^s)} n) \leq 2^{k+s} \epsilon^{1/((k-s)(k-s+1)2^s)}.$$

as desired. Thus, we may assume  $|S_1| \geq 2\epsilon^{1/(k-s+1)}n$ .

Similarly, it can be shown that for  $i = 1, \dots, k - s$ , there are subsets  $S_j$ , with  $j \leq i$ ,  $|S_j| \geq 2\epsilon^{1/(k-s+1)}n$  such that for  $\bar{w}_i = (w_1, \dots, w_i)$  with  $w_j \in S_j$  for all  $j \leq i$ , we have  $\text{dev}_s H_{\bar{w}_i} \geq (2^{k+s} - 2^i) \epsilon^{i/((k-s)(k-s+1)2^s)}$ . In particular, for  $w = (w_1, \dots, w_{k-s})$  with  $w_i \in S_i$ ,  $1 \leq i \leq k - s$ , we have  $\text{dev}_s H_w \geq (2^{k+s} - 2^{k-s}) \epsilon^{1/2^{(k-s+1)2^s}}$ .

For each  $w \in S_1 \times S_2 \times \dots \times S_{k-s}$ , the induction hypothesis implies that there exists a  $(s - 1)$ -graph  $G_w$  on  $V$  satisfying

$$|e(H_w, G_w) - e(\bar{H}_w, G_w)| > \delta n^s$$

where  $\delta$  satisfies  $\delta = 4^{-s} (2^{k+s-1})^{2^s} \epsilon^{1/(k-s+1)}$ .

Thus, there are subsets  $S'_i$  of  $S_i$ ,  $1 \leq i \leq k-s$ , with  $|S'_i| = \epsilon^{1/(k-s+1)}n$  so that for  $w \in S' = S'_1 \times S'_2 \times \dots \times S'_{k-s}$  either  
(a)  $e(H_w, G_w) - e(\bar{H}_w, G_w) \geq \delta n^s$  for all  $w \in S'$ ; or  
(b)  $e(H_w, G_w) - e(\bar{H}_w, G_w) \leq -\delta n^s$  for all  $w \in S'$ .

We will treat case (a) and omit the similar treatment for case (b).

We proceed to define the following  $(k-1)$ -graph  $G$  on  $V$ .

$$E(G) = \{w \cup y : y \in E(G_w)\} \setminus \binom{V \setminus (S'_1 \cup \dots \cup S'_{k-s})}{s-1}$$

For each  $x \in E_s(H, G)$ , there are three possibilities:

- (i)  $x$  contains more than one vertex in some  $S'_i$ . There are at most  $\epsilon n^k$  such edges.
- (ii)  $x$  has no vertex in  $S'_i$  for some  $i$ . In this case,  $x$  can not contain any edge in  $G$ , contradicting  $x \in E_s(H, G)$ .
- (iii)  $x$  has exactly one vertex  $w_i$  in  $S'_i$  for  $i = 1, \dots, k-s$ . Say,  $x = w \cup x'$ , where for any vertex  $u \in x'$  we have  $x \setminus \{u\} \in E(G)$ . Therefore, we have

$$\begin{aligned} E_s(H, G) - E_s(\bar{H}_G) &\geq \sum_{w \in S'} (E(H_w, G_w) - E(\bar{H}_w, G_w)) - \epsilon n^k \\ &\geq \prod_{i=1}^{k-s} |S'_i| \cdot \delta n^s - \epsilon n^k \\ &\geq \epsilon^{(k-s)/(k-s+1)} \cdot 3\epsilon^{1/(k-s+1)} n^k - \epsilon n^k \\ &\geq \epsilon n^k \end{aligned}$$

which is a contradiction. This completes the proof for (2).

Combining the above two theorem, we see that  $\text{dev}_s$  and  $\text{disc}^{(s)}$  are equivalent.

## 5 Concluding remarks

In a  $k$ -graph  $H$ , many questions can be asked concerning the  $(l, s)$ -discrepancy properties  $\text{disc}_l^{(s)}$ . For example, we have, for  $s \geq 3$ ,

$$\text{disc}_k^{(s)} \Rightarrow \text{disc}_k^{(s-1)} \tag{3}$$

by using the fact that  $\text{dev}_l \Rightarrow \text{dev}_{l-1}$  and the main theorem  $\text{disc}_k^{(s)} \Leftrightarrow \text{dev}_s$ . However, in a  $k$ -graph and  $2 \leq l < k$ , is it true that

$$\text{disc}_l^{(s)} \Rightarrow \text{disc}_l^{(s-1)}? \tag{4}$$

In the implication (3), the reversed direction does not hold (see [2]). For a general  $l$  with  $l < k$ , is it still true? Is it possible to have one equivalence class which includes  $\text{disc}_l^{(s)}$  for some consecutive values of  $s$  for some  $l$ ? What is then the length of the chain of equivalence classes containing  $\text{disc}_l^{(s)}$  as  $s$  ranges from 1 to  $\binom{k}{l-1}$ ?

Recall that  $\text{disc}_l = \text{disc}_l^{(s)}$  with  $s = \binom{k}{l-1}$ . From the definition, it is not hard to check that  $\text{disc}_l \Rightarrow \text{disc}_{l-1}$  for  $l \geq 3$ . To further explore the relations among  $\text{disc}_l^{(s)}$  and  $\text{disc}_{l-1}^{(t)}$ , we need more definitions.

We consider a  $k$ -graph  $H$  with vertex set  $V = V(H)$  and  $E = E(H)$ . Let  $Q$  denote a fixed  $l$ -graph on  $k$  vertices and  $G$  denote a  $l$ -graph on vertex set  $V$ . In a  $k$ -graph  $H$ , an edge  $x$  in  $E(H)$  is said to be  $Q$ -induced by  $G$  if there is an embedding  $\pi$  of  $Q$  into  $\binom{x}{l}$ , the set of  $l$ -subsets of  $x$  satisfying the property that for all  $y \in E(Q)$ , the images  $\pi(y)$  are in  $E(G)$ . Let  $e_Q(H, G)$  denote the total number of edges in  $H$  which are  $Q$ -induced by  $G$ .

**Definition.** For a  $k$ -graph  $H$  on vertex set  $V$  with  $|V| = n$  and a fixed  $(l-1)$ -graph  $Q$ , we define  $\text{disc}_l^Q H$  by:

$$\text{disc}_l^Q H = \frac{1}{n^k} \max_M |e_Q(H, G) - e_Q(\bar{H}, G)|,$$

where the maximum is taken over all  $(l-1)$ -graphs  $G$  on  $V$ .

It is of interest to examine possible necessary and sufficient conditions for a pair of graphs  $Q$  and  $Q'$  on  $k$  vertices such that  $\text{disc}_l^Q \Leftrightarrow \text{disc}_l^{Q'}$ . For a fixed  $k$  and a graph  $Q$ , how large is the family of graphs consisting of  $Q'$  satisfying the property that  $\text{disc}_l^{Q'}$  is in the quasi-random class that includes  $\text{disc}_l^Q$ ? How are properties  $\text{disc}_l^Q$  related to  $\text{disc}_l^{(s)}$ ? Most of all, what is the lattice structure illustrating the relations among quasi-random classes of  $k$ -graphs? In this note, we only example some very special parts of this lattice. Numerous questions remain to be explored.

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