

QUASI-RANDOM GRAPHS

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We introduce a large equivalence class of graph properties, all of which are shared by so-called random graphs. Unlike random graphs, however, it is often relatively easy to verify that a particular family of graphs possesses some property in this class.

1. Introduction

Perhaps the simplest model of generating a "random" graph G on n vertices is the process which considers each of the possible pairs $\{v, v'\}$ of vertices of G , and decides independently with probability $1/2$ whether or not $\{v, v'\}$ is an edge. Strictly speaking, this process induces a probability distribution on the space $\mathcal{S}(n)$ of (ordered) graphs on n vertices, with each particular graph having probability $2^{-\binom{n}{2}}$. It may happen that for some graph property \mathcal{P} , it is true that

$$\Pr \{G \in \mathcal{S}(n) : G \text{ satisfies } \mathcal{P}\} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

In this case, a typical graph in $\mathcal{S}(n)$, which we denote by $G_{1/2}(n)$, will have property \mathcal{P} with overwhelming probability as n gets large. (For a much fuller discussion of these concepts, the reader can consult [6], [15] or [2].) We sometimes abbreviate this by saying that $G_{1/2}(n)$ almost always has property \mathcal{P} . For example, $G_{1/2}(n)$ almost always has all but $o(n)$ of its vertices with degree $(1+o(1))n/2$.

The main thrust of this work will be to show the *equivalence* of a number of disparate graph properties, all possessed by almost all $G_{1/2}(n)$, in the following sense: Any graph satisfying *any one* of the properties must of necessity satisfy *all* the others. We term such graphs *quasi-random*. We follow much in the spirit of the recent seminal paper of Thomason [18] in which many properties of " (p, α) -jumbled" graphs are presented (see also [17]). In both cases, such graphs share many large scale properties with random graphs (with the appropriate edge probabilities). For ease of exposition we have restricted our attention here to quasi-random graphs corresponding to edge probability $1/2$ (at the end of the paper we mention the more general situation). Our initial impetus for this work had its roots in some early papers of Wilson [20], [21], and a more recent one of Rödl [16].

2. Notation

Let $G=(V, E)$ denote a graph with vertex set V and edge set E . We use the notation $G(n)$ (and $G(n, e)$) to denote that G has n vertices (and e edges). For $X \subseteq V$, we let $X|_G$ denote the subgraph of G induced by X , and we let $e(X)$ denote the number of edges of $X|_G$. For $v \in V$, define

$$nd(v) := \{x \in V: \{v, x\} \in E\}, \quad deg(v) := |nd(v)|.$$

Further, if $G'=(V', E')$ is another graph, we let $N_G^*(G')$ denote the number of (labelled) occurrences of G' as an induced subgraph of G . In other words, $N_G^*(G')$ is the number of injections $\lambda: V' \rightarrow V$ such that $\lambda(V')|_G \cong G'$. The quantity $N_G^*(G')$ is related to $\bar{N}_G^*(G')$, the number of unlabelled occurrences of G' in G by

$$N_G^*(G') = \bar{N}_G^*(G')/|Aut(G')|,$$

where $Aut(G)$ denotes the automorphism group of G . We will often just write $N^*(G')$ if G is understood. The final related notation we need is $N_G(G')$, the number of occurrences of G' as a (not necessarily induced) subgraph of G . Thus, if $G'=(V', E')$ then

$$(0) \quad N_G(G') = \sum_H N_G^*(H)$$

where the sum is taken over all $H=(V', E_H)$ where $E_H \supseteq E'$.

3. The main results

We next list a set of graph properties which a graph $G=G(n)$ might satisfy. Each of the properties will contain occurrences of the asymptotic "little-oh" notation $o(\cdot)$. However, the dependence of different $o(\cdot)$'s on the particular properties they refer to will ordinarily be suppressed. The use of these $o(\cdot)$'s can be viewed in two essentially equivalent ways.

In the first way, suppose we have two properties P and P' , each with occurrences of $o(1)$, say. Thus, $P=P(o(1))$, $P'=P'(o(1))$. The implication " $P \Rightarrow P'$ " then means that if each $o(1)$ in $P(o(1))$ is replaced by a fixed (but arbitrary) function $f(n)=o(1)$ (i.e., $f(n) \rightarrow 0$ as $n \rightarrow \infty$), then there is some other function $f'(n)=o(1)$ (depending on f) so that if $G(n)$ satisfies $P(f(n))$ then it must also satisfy $P'(f'(n))$. The particular choice made for f depends on the context, common ones being $n^{-1/2}$ and $1/\log n$ (when $f(n)=o(1)$).

In the second way, we can think of considering a family \mathcal{F} of graphs $G(n)$ with $n \rightarrow \infty$. In this case, the interpretation of $o(1)$ is the usual one as $G=G(n)$ ranges over \mathcal{F} .

$P_1(s)$: For all graphs $M(s)$ on s vertices,

$$N_G^*(M(s)) = (1 + o(1))n^s 2^{-\binom{s}{2}}.$$

The content of $P_1(s)$ is that all of the $2^{\binom{s}{2}}$ labelled graphs $M(s)$ on s vertices occur asymptotically the same number of times in G (just as we would expect for $G_{1/2}(n)$).

Let C_t denote the cycle with t edges.

$$P_2(t): e(G) \cong (1+o(1))\frac{n^2}{4}, \quad N_G(C_t) \cong (1+o(1))\left(\frac{n}{2}\right)^t.$$

Let $A = A(G) = (a(v, v'))_{v, v' \in V}$ denote the adjacency matrix of G , i.e., $a(v, v') = 1$ if $\{v, v'\} \in E$, and 0 otherwise. Order the eigenvalues λ_i of A (which of course are real) so that $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$.

$$P_3: e(G) \cong (1+o(1))\frac{n^2}{4}, \quad \lambda_1 = (1+o(1))\frac{n}{2}, \quad \lambda_2 = o(n).$$

We remark here that a result of Juhász [13] (see also [10]) shows that the random graph $G_{1/2}(n)$ has $\lambda_1 = (1+o(1))n/2$ and $\lambda_2 = o(n^{1/2+\epsilon})$ for any fixed $\epsilon > 0$.

$$P_4: \text{For each subset } S \subseteq V, \quad e(S) = \frac{1}{4}|S|^2 + o(n^2).$$

$$P_5: \text{For each subset } S \subseteq V \text{ with } |S| = \left\lfloor \frac{n}{2} \right\rfloor, \quad e(S) = \left(\frac{1}{16} + o(1) \right) n^2.$$

For $v, v' \in V$, define

$$s(v, v') := \{y \in V: a(v, y) = a(v', y)\}.$$

$$P_6: \sum_{v, v'} \left| s(v, v') - \frac{n}{2} \right| = o(n^3).$$

$$P_7: \sum_{v, v'} \left| |nd(v) \cap nd(v')| - \frac{n}{4} \right| = o(n^3).$$

There are several implications among the P_i which are immediate, e.g., $P_1(s) \Rightarrow P_2(s)$ and $P_4 \Rightarrow P_5$. Our main result asserts that for $s \geq 4$, and even $t \geq 0$, all the properties are in fact equivalent.

Theorem 1. For $s \geq 4$ and even $t \geq 4$,

$$P_2(4) \Rightarrow P_2(t) \Rightarrow P_1(s) \Rightarrow P_3 \Rightarrow P_4 \Rightarrow P_5 \Rightarrow P_6 \Rightarrow P_7 \Rightarrow P_2(4).$$

What was (initially) the most surprising to the authors was how strong the (apparently weak) condition $P_2(4)$ actually is. Graphs which satisfy any (and therefore, all) of these properties we call *quasi-random*.

A weaker property of $G(n)$ is the following.

$$P_0: \sum_v \left| \deg(v) - \frac{n}{2} \right| = o(n^2).$$

It follows easily (using the Cauchy—Schwarz inequality) that the following property is equivalent to P_0 :

$$P'_0: \text{All but } o(n) \text{ vertices of } G \text{ have degree } (1+o(1))\frac{n}{2}.$$

In this case we say that G is “almost-regular”.

Theorem 2.

$$P_1(4) \Rightarrow P_1(3) \Rightarrow P_0.$$

One immediate consequence of Theorem 1 is the following.

Corollary. *Let $\varepsilon > 0$ and suppose $G = G(n) = (V, E)$ is quasi-random. Then for any $X \subseteq V$ with $|X| > \varepsilon n$, the induced subgraph $X|_G$ is quasi-random.*

A number of results with a similar flavor have appeared in the literature. For example, in addition to the work of Thomason mentioned earlier (who, for example, proved $P_0 + P_7 \Rightarrow P_4$), Frankl, Rödl and Wilson [8] (extending earlier work of Rödl [16]) showed that $P_8 \Rightarrow P_1(s)$. (The proof given here is somewhat more direct.) In [1], Alon and Chung proved that for regular graphs, $P_3 \Rightarrow P_4$.

The flowchart shown in Fig. 1 gives an outline of our proof. The symbol F_i by an edge indicates that the corresponding implication is proved in Fact i .

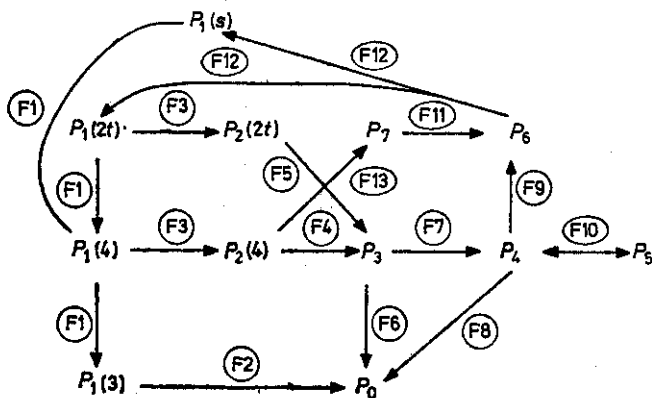


Fig. 1

In the final section we list various examples, counterexamples, extensions and open problems.

4. The proofs

We first make the following observation.

Fact 1. $P_1(s+1) \Rightarrow P_1(s)$.

Proof. Suppose $M(s)$ is a fixed graph on s vertices. There are 2^s ways to extend it to a graph on $s+1$ vertices. By $P_1(s+1)$, for each $(s+1)$ -vertex graph $M(s+1)$ we have

$$N_G^*(M(s+1)) = (1 + o(1))n^{s+1}2^{-\binom{s+1}{2}}.$$

Since each copy of $M(s)$ in G is contained in $n-s$ $(s+1)$ -vertex subgraphs $M(s+1)$, we obtain

$$N_G^*(M(s)) = (1 + o(1))n^{s+1}2^{-\binom{s+1}{2}}2^s/n = (1 + o(1))n^s2^{-\binom{s}{2}}$$

which is $P_1(s)$, as required. ■

Fact 2. $P_1(3) \Rightarrow P_0$.

Proof. Let H_i , $i=1, 2, 3$, denote graphs with 3 vertices and i edges. Then we have

$$(1) \quad \sum_v \text{deg}(v)(\text{deg}(v)-1) = N^*(H_2) + N^*(H_3) = (1 + o(1)) \frac{n^3}{4}$$

by $P_1(3)$ where $N^*(\cdot) = N_G^*(\cdot)$. On the other hand, by counting how often each edge can contribute to the various $N^*(H_i)$, we obtain

$$(2) \quad (n-2) \sum_v \text{deg}(v) = N^*(H_1) + 2N^*(H_2) + N^*(H_3) = (1 + o(1)) \frac{n^3}{2}.$$

Thus, we have by Cauchy—Schwarz,

$$(3) \quad (1 + o(1)) \frac{n^3}{4} \cong \sum_v (\text{deg}(v))^2 \cong \frac{1}{n} (\sum_v \text{deg}(v))^2 \cong (1 + o(1)) \frac{n^3}{4},$$

so that

$$\sum_v \text{deg}(v) = (1 + o(1)) \frac{n^2}{2}$$

which implies P_0 (or equivalently, P_0'). ■

Fact 3. $P_1(2t) \Rightarrow P_2(2t)$, $t \geq 2$.

Proof. This follows at once using (0) and Facts 1 and 2. ■

Fact 4. $P_2(4) \Rightarrow P_3$.

Proof. Denote the eigenvalues of $A = A(G)$ by

$$\lambda_1, \lambda_2, \dots, \lambda_n \quad \text{with} \quad |\lambda_1| \cong |\lambda_2| \cong \dots \cong |\lambda_n|.$$

First, it is easy to see that

$$(4) \quad |\lambda_1| \cong (1 + o(1)) \frac{n}{2}$$

since, for $\bar{v} = (1, 1, \dots, 1)^t$, we have (see [11])

$$(5) \quad |\lambda_1| \cong \frac{\langle A\bar{v}, \bar{v} \rangle}{\langle \bar{v}, \bar{v} \rangle} = \frac{1}{n} \sum_v \text{deg}(v) \cong \left(\frac{1}{2} + o(1) \right) n.$$

Next, consider the trace $tr(A^4)$ of A^4 . Clearly,

$$(6) \quad tr(A^4) = \sum_{i=1}^n \lambda_i^4 \cong |\lambda_1|^4 \cong (1 + o(1)) \frac{n^4}{16}.$$

On the other hand, by examining how terms can contribute to $tr(A^4)$ it is not hard to see that

$$(7) \quad tr(A^4) = N_G(C_4) + o(n^4) \cong (1 + o(1)) \frac{n^4}{16}.$$

Thus,

$$tr(A^4) = (1 + o(1)) \frac{n^4}{16}$$

which by (4) and (6) implies

$$(8) \quad \lambda_1 = (1 + o(1)) \frac{n}{2}$$

and, since all the λ_i are real,

$$\sum_{i=2}^n |\lambda_i|^4 = o(n^4)$$

so that $|\lambda_2| = o(n)$, as required. ■

Fact 5. $P_2(2t) \Rightarrow P_3, t \geq 2$.

The proof of Fact 5 is similar to that of Fact 4 and is omitted. It should be noted here that the distinction between even and odd values of u for $P_2(u)$ arises from the fact that when $u=2t$ is even, each of the individual terms λ_i^{2t} in the expression for $tr(A^{2t})$ is nonnegative, thus allowing bounds on their magnitudes to be derived from bounds on their sums (of $2t^{\text{th}}$ powers). This is not the case if u is odd, and indeed, we will give examples (Sec. 5) of graphs satisfying $P_2(2t+1)$ which are *not* quasi-random.

Fact 6. $P_3 \Rightarrow P_0$.

Proof. Let $\bar{v} = (1, 1, \dots, 1)^t$. Since

$$\|A\bar{v}\| \cong \lambda_1 \|\bar{v}\|$$

we have

$$\sum_v (\text{deg}(v))^2 \cong (1 + o(1)) \frac{n^3}{4}.$$

However, by assumption,

$$e(G) = \frac{1}{2} \sum_v \text{deg}(v) \cong (1 + o(1)) \frac{n^2}{4}$$

so that by Cauchy—Schwarz (as in (3))

$$\sum_v \left| \text{deg}(v) - \frac{n}{2} \right| = o(n^2)$$

as required. ■

Fact 7. $P_3 \Rightarrow P_4$.

Proof. Let \bar{e}_i denote a set of orthonormal eigenvectors corresponding to the eigenvalues λ_i of A (so that $\|\bar{e}_i\| = 1$). By hypothesis,

$$\lambda_1 = \left(\frac{1}{2} + o(1) \right) n, \quad \lambda_i = o(n), \quad i > 1.$$

Define $\bar{u} = \frac{1}{\sqrt{n}} (1, 1, \dots, 1)^t$.

Claim. $\|\bar{u} - \bar{e}_1\| = o(1)$.

Proof of Claim. Write $\bar{u} = \sum_i a_i \bar{e}_i$. Thus,

$$(9) \quad A\bar{u} = \sum_i a_i \lambda_i \bar{e}_i.$$

On the other hand, the j^{th} component of the vector $A\bar{u}$ is just $\text{deg}(v_j)/\sqrt{n}$ where v_j is the j^{th} vertex of G . Thus, by Fact 6, all but $o(n)$ components of $A\bar{u}$ are $\left(\frac{1}{2} + o(1)\right)\sqrt{n}$. This means we can write

$$A\bar{u} = \left(\frac{1}{2} + o(1)\right)n\bar{u} + \bar{w}$$

where all but $o(n)$ components of \bar{w} are $o(\sqrt{n})$, and so $\|\bar{w}\| = o(n)$. By (9) this implies

$$\sum_{i=1}^n \left(\lambda_i - \frac{n}{2}\right) a_i \bar{e}_i = \bar{w} + \bar{u} \cdot o(n),$$

$$\left(\sum_{i=1}^n \left(\lambda_i - \frac{n}{2}\right)^2 a_i^2\right)^{1/2} = \|\bar{w} + \bar{u} \cdot o(n)\| = o(n),$$

which in turn implies

$$\sum_{i=1}^n a_i^2 = o(1).$$

Since $\bar{u} = a_1 \bar{e}_1 + \bar{w}_1$ with $\|\bar{w}_1\| = o(1)$ while $\|\bar{u}\| = \|\bar{e}_1\| = 1$, we have $|a_1| = 1 + o(1)$. By a well known theorem of Frobenius (see [11]), all the coefficients of \bar{e}_1 (which is associated to λ_1 , the dominant eigenvalue of A) are nonnegative. Thus, $a_1 = 1 + o(1)$ which proves the Claim. ■

Proof of Fact 7. Let $\bar{s} = (s_1, \dots, s_n)$ be the characteristic vector of $S \subseteq V = \{v_1, \dots, v_n\}$, i.e.,

$$s_i = \begin{cases} 1 & \text{if } v_i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

We want to show

$$(10) \quad e(S) = \frac{1}{4} |S|^2 + o(n^2).$$

Define $\bar{s}' = \bar{s} - \langle \bar{s}, \bar{e}_1 \rangle \bar{e}_1$. Since $\langle \bar{s}', \bar{e}_1 \rangle = 0$ then

$$(11) \quad \langle A\bar{s}', \bar{s}' \rangle \leq |\lambda_2| \|\bar{s}'\|^2.$$

We will next estimate $\|\bar{s}'\|^2$ and $\langle A\bar{s}', \bar{s}' \rangle$. First, we have

$$(12) \quad \|\bar{s}'\|^2 = \|\bar{s} - \langle \bar{s}, \bar{e}_1 \rangle \bar{e}_1\|^2 \leq \|\bar{s}\|^2 = |S|.$$

Also,

$$(13) \quad \langle A\bar{s}', \bar{s}' \rangle = \langle A\bar{s}, \bar{s} \rangle - 2\langle \bar{s}, \bar{e}_1 \rangle \langle A\bar{s}, \bar{e}_1 \rangle + \langle \bar{s}, \bar{e}_1 \rangle^2 \langle A\bar{e}_1, \bar{e}_1 \rangle = 2e(S) - \lambda_1 \langle \bar{s}, \bar{e}_1 \rangle^2$$

and

$$\langle \bar{s}, \bar{e}_1 \rangle = \langle \bar{s}, \bar{u} + \bar{w}_1 \rangle = |S|/\sqrt{n} + \langle \bar{s}, \bar{w}_1 \rangle.$$

By the Claim, $\|\bar{w}_1\| = o(1)$, so that

$$|\langle \bar{s}, \bar{w}_1 \rangle| \leq \|\bar{s}\| \|\bar{w}_1\| = \sqrt{o(|S|)}.$$

Thus,

$$(14) \quad \langle \bar{s}, \bar{e}_1 \rangle = |S|/\sqrt{n} + o(\sqrt{|S|}).$$

Therefore, by (11), (12) and (13),

$$\langle AS', \bar{s}' \rangle = 2e(S) - \left(\frac{1}{2} + o(1) \right) |S|^2 + o(n^2) \cong \lambda_2 \| \bar{s}' \|^2 \cong |S| \cdot o(n).$$

This implies

$$e(S) = \frac{1}{4} |S|^2 + o(n^2)$$

as required, and the proof of Fact 7 is completed. ■

Fact 8. $P_4 = P_0$.

Proof. Suppose for any subset $S \subseteq V$

$$(15) \quad \left| e(S) - \frac{1}{4} |S|^2 \right| < \frac{\varepsilon^2 n^2}{3}.$$

We will show that fewer than εn vertices of G have degree greater than $\left(\frac{1}{2} + \varepsilon \right) n$. Suppose to the contrary that there is a set T of $t \geq \varepsilon n$ vertices of degree greater than $\left(\frac{1}{2} + \varepsilon \right) n$. Thus,

$$\sum_{v \in T} \deg(v) \geq \left(\frac{1}{2} + \varepsilon \right) tn.$$

By hypothesis,

$$e(G) < \frac{n^2}{4} + \frac{\varepsilon^2 n^2}{3}$$

$$e(T) < \frac{t^2}{4} + \frac{\varepsilon^2 n^2}{3}$$

$$e(T^c) > \frac{(t')^2}{4} - \frac{\varepsilon^2 n^2}{3}$$

where $T^c := V \setminus T$ and $t' := |T^c|$. Since

$$(16) \quad e(T^c) + \sum_{v \in T} \deg(v) = e(G) + e(T)$$

then by the preceding estimates,

$$(17) \quad \frac{(t')^2}{4} - \frac{\varepsilon^2 n^2}{3} + \left(\frac{1}{2} + \varepsilon \right) tn < \frac{n^2}{4} + \frac{\varepsilon^2 n^2}{3} + \frac{t^2}{4} + \frac{\varepsilon^2 n^2}{3}.$$

This in turn implies

$$(18) \quad \varepsilon tn < \varepsilon^2 n^2$$

which is impossible for $t \geq \varepsilon n$.

It follows in the same way that fewer than εn vertices of G have degree less than $\left(\frac{1}{2} - \varepsilon \right) n$. This implies P_0 , and the Fact is proved. ■

Fact 9. $P_4 \Rightarrow P_6$. Assume that for any $S \subseteq V$,

$$\left| e(S) - \frac{1}{4} |S|^2 \right| < \frac{\varepsilon^2 n^2}{3}.$$

We will show that

$$\sum_{v, v'} \left| s(v, v') - \frac{n}{2} \right| < 20\varepsilon n^2,$$

which will imply P_6 .

From the proof of Fact 8, all vertices of V except for a set Y of size at most $2\varepsilon n$ have degrees between $\left(\frac{1}{2} - \varepsilon\right)n$ and $\left(\frac{1}{2} + \varepsilon\right)n$. For vertices $v, v' \in V$, define

$$f_{ij}(v, v') := |\{w \in V: a(v, w) = i, a(v', w) = j\}|$$

for $0 \leq i, j \leq 1$. Thus,

$$\left| f_{ij}(v, v') + f_{i'j'}(v, v') - \frac{n}{2} \right| \leq \varepsilon n$$

if $v, v' \in V \setminus Y := V'$ and, $(i, j) = (0, 0)$ or $(1, 1)$, and $(i', j') = (1, 0)$ or $(0, 1)$. Thus, in this case

$$|f_{11}(v, v') - f_{00}(v, v')| \leq 2\varepsilon n.$$

For a fixed $v \in V'$, let $X(v)$ denote the set

$$\left\{ v' \in V': \left| s(v, v') - \frac{n}{2} \right| > 10\varepsilon n \right\}.$$

There are two possibilities:

(i) For all $v \in V'$, $|X(v)| \leq 2\varepsilon n$. Thus,

$$\sum_{v, v'} \left| s(v, v') - \frac{n}{2} \right| \leq 20\varepsilon n^2$$

and we are done.

(ii) For some $v_0 \in V'$, $|X(v_0)| > 2\varepsilon n$. Define

$$X_1 = \left\{ u \in X(v_0): s(v_0, u) > \frac{n}{2} + 10\varepsilon n \right\},$$

$$X_2 = \left\{ u \in X(v_0): s(v_0, u) < \frac{n}{2} - 10\varepsilon n \right\}.$$

Since $|X_1| + |X_2| = |X(v_0)|$ then either $|X_1| \geq \varepsilon n$ or $|X_2| \geq \varepsilon n$. We will treat the former case; the argument for the latter is very similar and is omitted. Now, for each $v \in X_1$, v is adjacent to $f_{11}(v_0, v)$ vertices in $nd(v_0)$. Since $s(v_0, v) > \frac{n}{2} + 10\varepsilon n$, we get

$$f_{11}(v_0, v) \geq (s(v_0, v) - 2\varepsilon n) \cdot 1/2 \geq \frac{n}{4} + 4\varepsilon n.$$

Thus, the number of edges $e(X_1, nd(v_0))$, which is the number of ordered pairs (u, v) , $u \in X_1$, $v \in nd(v_0)$, is at least $|X_1| \left(\frac{n}{4} + 4\epsilon n \right)$. By hypothesis,

$$e(X_1) > \left(\frac{1}{4} |X_1|^2 \right) - \frac{\epsilon^2 n^2}{3}, \quad e(nd(v_0)) > \left(\frac{1}{4} |nd(v_0)|^2 \right) - \frac{\epsilon^2 n^2}{3},$$

$$e(X_1 \cap nd(v_0)) < \frac{1}{4} |X_1 \cap nd(v_0)|^2 + \frac{\epsilon^2 n^2}{3}.$$

Thus,

$$e(X_1 \cup nd(v_0)) \cong e(X_1) + e(nd(v_0)) + |X_1| \left(\frac{n}{4} + 4\epsilon n \right) - 3e(X_1 \cap nd(v_0)) \cong$$

$$(19) \quad \cong \frac{1}{4} |X_1|^2 + \frac{1}{4} |nd(v_0)|^2 + |X_1| \left(\frac{n}{4} + 4\epsilon n \right) - \frac{5\epsilon^2 n^2}{3} - \frac{3}{4} |X_1 \cap nd(v_0)|^2 \cong \\ \cong \frac{1}{4} |X_1 \cup nd(v_0)|^2 + \epsilon^2 n^2.$$

However, by hypothesis

$$(20) \quad e(X_1 \cup nd(v_0)) \cong \frac{1}{4} |X_1 \cup nd(v_0)|^2 + \frac{\epsilon^2 n^2}{3}.$$

This is a contradiction to (19). This completes the proof of Fact 9. ■

Fact 10. $P_3 \Rightarrow P_4$.

Proof. Suppose that for any subset $S \subseteq V$ with $|S| = \lfloor n/2 \rfloor$,

$$\left| e(S) - \frac{n^2}{16} \right| < \epsilon n^2$$

where ϵ is fixed. We will show that for any $S' \subseteq V$

$$\left| e(S') - \frac{1}{2} \binom{|S'|}{2} \right| < 20\epsilon n^2.$$

We will consider two cases:

(i) $|S'| \cong n/2$. By averaging over all subsets S'' of S' of size $\lfloor n/2 \rfloor$ we have

(21)

$$e(S') = \sum_{S'' \subseteq S'} e(S'') / \binom{|S'| - 2}{\lfloor n/2 \rfloor - 2} \cong \frac{|S'|(|S'| - 1)}{\lfloor n/2 \rfloor (\lfloor n/2 \rfloor - 1)} \left(\frac{n^2}{16} + \epsilon n^2 \right) \cong \binom{|S'|}{2} \left(\frac{1}{2} + 8\epsilon \right).$$

In the same way we can prove the corresponding lower bound

$$e(S') \cong \binom{|S'|}{2} \left(\frac{1}{2} - 8\epsilon \right)$$

and this case is completed.

(ii) $|S'| < \frac{n}{2}$. Suppose

$$(22) \quad e(S') \cong \frac{1}{2} \binom{|S'|}{2} + 20\epsilon n^2.$$

Since $n - |S'| > \frac{n}{2}$ than by case (i) for $\bar{S}' := V \setminus S'$ we have

$$(23) \quad \begin{aligned} e(\bar{S}') &> \binom{n - |S'|}{2} \left(\frac{1}{2} - 8\epsilon \right), \\ e(\bar{S}') &< \binom{n - |S'|}{2} \left(\frac{1}{2} + 8\epsilon \right). \end{aligned}$$

Thus, the number of edges $e(S', \bar{S}')$ between S' and \bar{S}' is

$$e(S', \bar{S}') = e(G) - e(S') - e(\bar{S}').$$

Now we will consider the average value of $e(S' \cup S'')$ where S'' ranges over all subsets of \bar{S}' of size $[n/2] - |S'|$ (so that $|S' \cup S''| = [n/2]$). This average is just

$$\begin{aligned} &\sum_{S'' \subseteq \bar{S}'} e(S' \cup S'') / \binom{[n/2] - |S'|}{[n/2] - |S'|} = \\ &= \binom{[n/2] - |S'|}{[n/2] - |S'|}^{-1} \left\{ e(S') \binom{[n/2] - |S'|}{[n/2] - |S'|} + e(\bar{S}') \binom{[n/2] - |S'| - 2}{[n/2] - |S'| - 2} \right. \\ &\quad \left. + e(S', \bar{S}') \binom{[n/2] - |S'| - 1}{[n/2] - |S'| - 1} \right\} = \\ &= e(S') + \frac{([n/2] - |S'|)([n/2] - |S'| - 1)}{(n - |S'|)(n - |S'| - 1)} e(\bar{S}') + \frac{([n/2] - |S'|)}{(n - |S'|)} e(S', \bar{S}') = \\ &= \frac{[n/2]}{n - |S'|} e(S') - \frac{([n/2] - |S'|)[n/2]}{(n - |S'|)(n - |S'| - 1)} e(\bar{S}') + \frac{([n/2] - |S'|)}{(n - |S'|)} e(G) > \\ &> \frac{[n/2]}{n - |S'|} \left(\frac{1}{2} \binom{|S'|}{2} + 20\epsilon n^2 \right) - \frac{([n/2] - |S'|)[n/2]}{(n - |S'|)(n - |S'| - 1)} \binom{n - |S'|}{2} \left(\frac{1}{2} + 8\epsilon \right) + \\ &\quad + \frac{[n/2] - |S'|}{n - |S'|} \binom{n}{2} \left(\frac{1}{2} - 8\epsilon \right) > \frac{n^2}{16} + \epsilon n^2. \end{aligned}$$

However, this contradicts the hypothesis that all $X \subseteq V$ with $|X| = [n/2]$ have $e(X) < \left(\frac{1}{16} + \epsilon \right) n^2$. In the same way it follows that

$$e(S') > \frac{1}{2} \binom{|S'|}{2} - 20\epsilon n^2.$$

This completes the proof of Fact 10. The reverse implication $P_5 \Rightarrow P_4$ is immediate. ■

Fact 11. $P_7 \Rightarrow P_8$.

Proof. Let $A = A(G)$ denote the adjacency matrix of G , with eigenvalues λ_i where $|\lambda_1| \cong |\lambda_2| \cong \dots \cong |\lambda_n|$. Since all but $o(n^2)$ entries of $A^2 := (b(v, v'))_{v, v' \in V}$ are $(1 + o(1))n/4$ (by P_7) then for $\bar{v} := (1, 1, \dots, 1)^t$ we have

$$\lambda_1^2 \|\bar{v}\|^2 = \lambda_1^2 n \cong \|A\bar{v}\|^2 = \langle A\bar{v}, A\bar{v} \rangle = \langle A^2 \bar{v}, \bar{v} \rangle = (1 + o(1))n^3/4$$

i.e.,

$$\lambda_1 \cong (1 + o(1))n/2.$$

Since all the λ_i are real then

$$\text{tr}(A^4) = \sum_i \lambda_i^4 \cong \lambda_1^4 \cong (1 + o(1))n^4/16.$$

On the other hand,

$$\text{tr}(A^4) = \sum_{v, v'} b(v, v')b(v', v) = (1 + o(1))n^2(n/4)^2 = (1 + o(1))n^4/16$$

which implies

$$\lambda_1 = (1 + o(1))n/2, \quad \lambda_2 = o(n).$$

Now, define $\bar{u} := \bar{v}/\sqrt{n}$ and let $\bar{e}_1, \dots, \bar{e}_n$ denote a set of orthonormal eigenvectors for $\lambda_1, \dots, \lambda_n$. Writing $\bar{u} = \sum_i a_i \bar{e}_i$ we have

$$A^2 \bar{u} = \sum_i a_i \lambda_i^2 \bar{e}_i = (1 + o(1)) \frac{n^2}{4} \bar{u} + \bar{w} = (1 + o(1)) \frac{n^2}{4} \sum_i a_i \bar{e}_i + \bar{w}$$

where all but $o(n)$ components of \bar{w} are $o(n^{3/2})$. Thus,

$$\sum_{i>1} \left(\lambda_i^2 - \frac{n^2}{4} \right)^2 a_i^2 \cong \sum_i \left(\lambda_i^2 - \frac{n^2}{4} \right)^2 a_i^2 = \|\bar{w} + o(n^2)\bar{u}\|^2 = o(n^4)$$

which implies $\sum_{i>1} a_i^2 = o(1)$. Since $\bar{u} = a_1 \bar{e}_1 + \bar{w}_1$ with $\|\bar{w}_1\| = o(1)$ and $\|\bar{u}\| = 1 = \|\bar{e}_1\|$ then we have $a_1 = 1 + o(1)$. Therefore,

$$\langle A\bar{u}, \bar{u} \rangle = \frac{1}{n} \sum_v \text{deg}(v) = \langle A(\bar{e}_1 + \bar{w}_1), (\bar{e}_1 + \bar{w}_1) \rangle = (1 + o(1)) \frac{n}{2}$$

which implies $\sum_v \text{deg}(v) = (1 + o(1)) \frac{n^2}{2}$. Since

$$\sum_{v, v'} |nd(v) \cap nd(v')| = \sum_u \text{deg}(u)(\text{deg}(u) - 1) = (1 + o(1))n^3/4$$

then by Cauchy—Schwarz we see that G is almost regular, i.e., satisfies P_8 . How-

ever, this implies that almost all pairs v, v' have each $f_{ij}(v, v')$ (from Fact 9) equal to $(1+o(1))n/4$. This in turn clearly implies P_6 . ■

Fact 12. $P_6 \Rightarrow P_1(s)$.

Proof. Suppose

$$(24) \quad \sum_{v, v'} \left| s(v, v') - \frac{n}{2} \right| = o(n^3).$$

We will show that for any graph $M(s)$ on s vertices, the number $N_s := N_G^*(M(s))$ satisfies

$$N_s = (1 + o(1))n^s 2^{-\binom{s}{2}}.$$

Assume the vertex set of $M(s)$ is $\{v_1, v_2, \dots, v_s\}$. For $1 \leq r \leq s$, define $M(r)$ to be the subgraph of M induced by the vertex set $V_r := \{v_1, v_2, \dots, v_r\}$. We prove by induction on r that

$$(25) \quad N_r := N_G^*(M(r)) = (1 + o(1))n_{(r)} 2^{-\binom{r}{2}}$$

where

$$n_{(r)} := n(n-1) \dots (n-r+1).$$

For $r=1$, (25) is immediate. Assume for some r , $1 \leq r < s$, that (25) holds. Define $\alpha := (\alpha_1, \dots, \alpha_r)$ where the α_i are distinct elements of $[n] := \{1, 2, \dots, n\}$ (which we take to be the vertex set of G). Also define $\varepsilon := (\varepsilon_1, \dots, \varepsilon_r)$, $\varepsilon_i = 0$ or 1 , and (as usual) for $i, j \in [n]$, $a(i, j) = 1$ if $\{i, j\}$ is an edge of G , and 0 otherwise. Finally, define

$$f_r(\alpha, \varepsilon) = |\{i \in [n] : i \neq \alpha_1, \dots, \alpha_r \text{ and } a(i, \alpha_j) = \varepsilon_j, 1 \leq j \leq r\}|.$$

Note that N_{r-1} is the sum of exactly N_r quantities $f_r(\alpha, \varepsilon)$. Namely, for each embedding of $M(r)$ into G , say $\lambda(v_j) = \alpha_j$, $1 \leq j \leq r$, $f_r(\alpha, \varepsilon)$ counts the number of ways of choosing $i \in [n]$ so that if we extend λ to V_{r+1} by setting $\lambda(v_{r+1}) = i$, and take $\varepsilon_j = a(v_{r+1}, v_j)$, then λ becomes an embedding of $M(r+1)$ into G . Also note that there are just $n_{(r)} 2^r$ quantities $f_r(\alpha, \varepsilon)$, since there are $n_{(r)}$ choices for α and 2^r choices for ε . Our next step will be to compute the first and second moments of $f_r(\alpha, \varepsilon)$.

To begin with, we have

$$\bar{f}_r := \frac{1}{n_{(r)} 2^r} \sum_{\alpha, \varepsilon} f_r(\alpha, \varepsilon) = \frac{1}{n_{(r)} 2^r} \sum_{\alpha} \sum_{\varepsilon} f_r(\alpha, \varepsilon) = \frac{1}{n_{(r)} 2^r} \sum_{\alpha} (n-r) = \frac{n-r}{2^r}$$

since every vertex $i \neq \alpha_1, \dots, \alpha_r$ corresponds to a unique choice for ε . Thus,

$$\sum_{\alpha, \varepsilon} f_r(\alpha, \varepsilon) = (n-r)n_{(r)} = n_{(r+1)}.$$

Next, define

$$S_r := \sum_{\alpha, \varepsilon} f_r(\alpha, \varepsilon)(f_r(\alpha, \varepsilon) - 1).$$

We claim that

$$(26) \quad S_r = \sum_{i \neq j} s(i, j)_{(r)}.$$

To see this, we interpret S_r as counting the number of ways of choosing

$\alpha = (\alpha_1, \dots, \alpha_r)$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r)$ and two other (ordered) vertices i and j in $[n]$ so that

$$a(i, \alpha_k) = \varepsilon_k = a(j, \alpha_k), \quad 1 \leq k \leq r.$$

Summing over all possible ε reduces this to requiring just that

$$a(i, \alpha_k) = a(j, \alpha_k), \quad 1 \leq k \leq r.$$

Now, think of choosing i and j first. The required additional r vertices $\alpha_1, \dots, \alpha_r$ must come exactly from $\{v \in [n] : a(i, v) = a(j, v)\}$. Therefore, there are $s(i, j)_{(r)}$ ways to choose them, which implies (26).

We next assert that (24) implies

$$(27) \quad \sum_{i \neq j} s(i, j)_{(r)} = (1 + o(1)) n^{r+2-2r}.$$

To see this, first define

$$\varepsilon_{ij} := s(i, j) - \frac{n}{2}.$$

By (24), $\sum_{i \neq j} |\varepsilon_{ij}| = o(n^2)$. Also, $|\varepsilon_{ij}| \leq n$. Thus,

$$\sum_{i \neq j} |\varepsilon_{ij}|^a \leq n^{a-1} \sum_{i \neq j} |\varepsilon_{ij}| = o(n^{a+2}), \quad a \text{ fixed.}$$

Therefore,

$$\begin{aligned} \sum_{i \neq j} s(i, j)_{(r)} &= \sum_{i \neq j} \left(\frac{n}{2} + \varepsilon_{ij} \right)_{(r)} = \\ &= \sum_{k=0}^r \sum_{i \neq j} c_k \left(\frac{n}{2} \right)^k \varepsilon_{ij}^{-k} \quad (\text{for appropriate constants } c_k) = \\ &= \left(\frac{n}{2} \right)^r n_{(2)} + \sum_{k=0}^{r-1} \sum_{i \neq j} c_k \left(\frac{n}{2} \right)^k \varepsilon_{ij}^{-k} \leq \\ &\leq \left(\frac{n}{2} \right)^r n_{(2)} + \sum_{k=0}^{r-1} \sum_{i \neq j} |c_k| \cdot |\varepsilon_{ij}|^{r-k} n^k \leq \\ &\leq \left(\frac{n}{2} \right)^r n_{(2)} + c \sum_{k=0}^{r-1} n^k \sum_{i \neq j} |\varepsilon_{ij}|^{r-k} \leq \\ &\leq \left(\frac{n}{2} \right)^r n_{(2)} + c \sum_{k=0}^{r-1} n^k \cdot o(n^{r-k+2}) = \\ &= \left(\frac{n}{2} \right)^r n_{(2)} + o(n^{r+2}) = \\ &= n^{r+2} 2^{-r} (1 + o(1)) \end{aligned}$$

as claimed.

Note that by (26) and (27) we have

$$(28) \quad S_r = (1 + o(1)) n^{r+2} 2^{-r}.$$

Consequently,

$$\sum_{\alpha, \varepsilon} (f_r(\alpha, \varepsilon) - \bar{f}_r)^2 = \sum_{\alpha, \varepsilon} f_r^2(\alpha, \varepsilon) - \sum_{\alpha, \varepsilon} \bar{f}_r^2 = \sum_{\alpha, \varepsilon} (f_r^2(\alpha, \varepsilon) - f_r(\alpha, \varepsilon)) + \\ + \sum_{\alpha, \varepsilon} f_r(\alpha, \varepsilon) - n_{(r)} 2^r (n-r)^2 2^{-2r} = S_r + n_{(r+1)} - n_{(r)}(n-r)^2 2^{-r} = o(n^{r+2}).$$

Finally, since from our earlier observation that

$$N_{r+1} = \sum_{\substack{N_r \text{ choices} \\ \text{of } (\alpha, \varepsilon)}} f_r(\alpha, \varepsilon)$$

then

$$|N_{r+1} - N_r \bar{f}_r|^2 = \left| \sum_{N_r \text{ terms}} (f_r(\alpha, \varepsilon) - \bar{f}_r) \right|^2 \cong \\ \cong N_r \sum_{N_r \text{ terms}} (f_r(\alpha, \varepsilon) - \bar{f}_r)^2 \quad \text{by Cauchy-Schwarz} \cong \\ \cong N_r \sum_{\alpha, \varepsilon} (f_r(\alpha, \varepsilon) - \bar{f}_r)^2 = \\ = o(N_r \cdot n^{r+2}) = o(n^{2r+2})$$

by induction. Consequently,

$$|N_{r+1} - N_r \bar{f}_r| = o(n^{r+1})$$

and so,

$$N_{r+1} = N_r \bar{f}_r + o(n^{r+1}) = (1 + o(1)) n_{(r)} 2^{-\binom{r}{2}} \cdot (n-r) 2^{-r} + o(n^{r+1}) = \\ = (1 + o(1)) n_{(r+1)} 2^{-\binom{r+1}{2}}.$$

This completes the induction step and Fact 12 is proved. ■

Fact 13. $P_2(r) \Rightarrow P_7$.

Proof. This follows at once from the observation that

$$N_G(C_4) = 2 \sum_{v, v'} |nd(v) \cap nd(v')|_{(2)} \cong (1 + o(1)) \frac{n^4}{16}.$$

Applying Cauchy-Schwarz (twice) now gives the desired conclusion. ■

5. Examples and counterexamples

In this section we present examples of quasi-random graphs as well as counterexamples to quasi-randomness for various weakened forms of the previous graph properties considered. We conclude with a discussion of open problems and future directions.

To begin with we mention one of the most widely used examples of a deterministic "random" graph, the so-called quadratic residue (or Paley) graph Q_p (e.g., see [3], [12], [17]). It is defined for a prime $p \equiv 1 \pmod{4}$ by choosing $\{i, j\}$ to be an edge of Q_p precisely when $i - j$ is a quadratic residue of p . It is common to rely on estimates of Weil [19] or Burgess [4] for character sums to establish the randomness properties of Q_p (see [12], [3]). However, it is quite easy to show that the quadratic residue graphs are quasi-random. To see this, observe that a vertex z is adjacent to both, or non-adjacent to both, of a pair x, y of distinct vertices of Q_p

if and only if the quotient $\frac{z-x}{z-y}$ is a quadratic residue of p . But for any of the $\frac{1}{2}(p-1)-1$ quadratic residues a other than 1, there is unique z such that

$$\frac{z-x}{z-y} = 1 + \frac{y-x}{z-y} = a.$$

Thus, $s(x, y) = \frac{1}{2}(p-3)$ so that P_6 holds.

We point out that Q_p does deviate from random graph $G_{1/2}(p)$ in the following way. The expected size of the largest clique in $G_{1/2}(p)$ has size $(1+o(1))(\log p)/(\log 2)$ (e.g., see [2]). However, the size of the largest clique in Q_p is now known by a recent result of S. Graham and C. Ringrose [23] to be as large as $c \log p \log \log \log p$ for infinitely many primes p . Earlier results of Montgomery [14] show that assuming the Generalized Riemann Hypothesis, we would have in fact a lower bound of $c \log p \log \log p$ infinitely often.

Yet another family of examples of quasi-random graphs arises from finite projective or affine planes. Let Π be an affine plane of order n , for example, a 2-dimensional vector space over a field of n elements. Let S be a subset of the $n+1$ points at infinity (i.e., a subset of the $n+1$ "slopes" of lines) and define a graph G_n whose vertices are the points of Π and where vertices x and y are adjacent if and only if the slope of the line of Π they determine belongs to the set S . As long as $|S| \approx \frac{n}{2}$, G_n will be quasi-random. The only property among P_1 — P_7 which is easily verified directly is P_6 ; the others follow, of course. (In fact, if $|S| = \frac{n+1}{2}$,

then $s(x, y)$ is exactly $\frac{1}{2}(n^2-3)$ for any pair of points x, y . For any S , G_n is strongly regular — these are examples of so-called Latin square graphs.)

Simple observations also show that the following graph $G(n)$ (or any of its many relatives; see [9], [7]) are quasi-random: The vertices of $G(n)$ are the n -sets of a fixed $2n$ -set; $\{v, v'\}$ is an edge of $G(n)$ iff $|v \cap v'| \equiv 0 \pmod{2}$.

By a *bisector* of a graph G on a set V of n vertices, we mean the set of edges between some set $X \subset V$ of size $\lfloor n/2 \rfloor$ and the complementary set $\bar{X} := V \setminus X$. In a random graph, we expect the number of edges $e(X, \bar{X})$ in any bisector to satisfy

$$(29) \quad e(X, \bar{X}) = \left(\frac{1}{8} + o(1) \right) n^2.$$

This also holds (by property P_6) for quasi-random graphs as well. However, having good bisectors is *not* enough to guarantee quasi-randomness as the following example shows.

Let $G(n)$ denote a graph on n vertices constructed as follows. The vertex set of $G(n)$ consists of two disjoint sets V and V' of sizes $\lfloor n/2 \rfloor$ and $\lfloor n/2 \rfloor$, respectively. On V we have a complete graph while on V' there are no edges. Between V and V' we place a random bipartite graph (with edge probability $1/2$). A simple computation shows that (29) holds for any set $X \subset V \cup V'$ with $|X| = \lfloor n/2 \rfloor$, although $G(n)$ is far from being quasi-random.

However, it is true that (29) together with almost-regularity (or property P_6) is in fact a quasi-random property.

Let $G^*(4n)$ denote a graph on $4n$ vertices constructed as follows. The vertex set of $G^*(4n)$ consists of four disjoint sets V_1, V_2, V_3, V_4 , each of size n . On V_1 and V_2 we have complete subgraphs. Between V_3 and V_4 we have a complete bipartite graph. Between $V_1 \cup V_2$ and $V_3 \cup V_4$ we place a random graph (with edge probability $1/2$). It is easy to check that $G^*(4n)$ satisfies $P_t(3)$, P_0 and $P_2(2t+1)$ for any fixed t , but is *not* quasi-random. As mentioned earlier, this shows the real difference there is in this context between even and odd cycles.

Let $H(n)$ be the graph consisting of the disjoint union of a complete graph $K_{n/2}$ and an independent set K_n of size $n/2$. Then $N_{H(n)}(C_4) = (1 + o(1)) \frac{n^4}{16}$ although $H(n)$ is not quasi-random. Of course, $H(n)$ fails to satisfy the edge constraint required for $P_4(4)$.

In a similar vein, the graph $L(n)$ consisting of a star with degree $n/2$ together with $n/2$ independent vertices has largest eigenvalue $\lambda_1 = (1 + o(1)) \frac{n}{2}$, and second largest eigenvalue $\lambda_2 = o(n)$, but is not quasi-random. Again the problem is the failure to satisfy the edge requirement of P_3 .

Let us call a family \mathcal{F} of graphs *forcing* if $N_{G(n)}(F) = (1 + o(1))n^e 2^{-e}$ for all $F = F(v, e) \in \mathcal{F}$ implies $G(n)$ is quasi-random. In other words, if each $F \in \mathcal{F}$ occurs as a subgraph of $G(n)$ about the same number of times it does in random graph $G_{1/2}(n)$ on n vertices, then $G(n)$ is quasi-random. What are the forcing families? For example, by Fact 4 we see that if P_t denotes the path t vertices then $\{P_2, C_4\}$ is a forcing family (as is $\{P_2, C_{2t}\}$, more generally). On the other hand, as we have just noted, $\{P_2, P_3, C_3\}$ is not a forcing family. Other examples of forcing families are $\{C_{2s}, C_{2t}\}$, $s \neq t$, $\{P_2, K_{2,t}\}$, $t \geq 2$, and $\{K_{2,s}, K_{2,t}\}$, $s \neq t \geq 2$.

Another variation of this is the following. Let us call a graph F with t distinguished vertices v_1, \dots, v_t *forcing* if:

$$(30) \quad \sum_{i_1, \dots, i_t} |s(i_1, \dots, i_t) - E(F)| = o(n^t) \cdot E(F)$$

(where $s(i_1, \dots, i_t)$ is the number of mappings λ of F into $G(n)$ with $\lambda(v_k) = i_k$, $1 \leq k \leq t$, and $E(F) = N_F(G_{1/2}(n))$, the expected number of occurrences of F in a random graph $G_{1/2}(n)$) implies $G(n)$ is quasi-random. It is not difficult to show that if F is any star with all endpoints distinguished, or a path of length 3 with both endpoints distinguished, or C_4 with two opposite vertices distinguished, then F is forcing. However, a triangle with one vertex specified is not forcing. This can be seen because for the graph G consisting of two identical disjoint components, each being a random graph $G_p(n/2)$ with $p = 2^{-1/3}$, (30) holds with $F = K_3$ (although G is not quasi-random). A challenging problem is to characterize the forcing graphs.

As we mentioned at the beginning we have restricted our notion of quasi-randomness to correspond to "imitating" the graphs $G_{1/2}(n)$. The analogous results can be established by basically the same arguments for the general case $G_p(n)$, $0 < p < 1$. In fact, many of the results can be extended to the case when $p = p(n) \rightarrow 0$ as $n \rightarrow \infty$ (e.g., $p(n) = n^{-\alpha}$ for various $\alpha > 0$). However, these investigations we leave for a later paper (see also [17]).

We point out here that the following interesting related question has been raised by Erdős and Hajnal in [5]. Suppose H is a fixed graph and $G(n)$ contains no induced subgraph isomorphic to H . Is it then necessarily true that either $G(n)$ or $\bar{G}(n)$ contains a complete subgraph of size n^ϵ for a fixed $\epsilon > 0$? It is known to be true if n^ϵ is replaced by $\exp(c \sqrt{\log n})$.

Finally, we mention that it would be of great interest to know what the analogues of the preceding results might be for hypergraphs. Some first steps in this direction are taken in [22].

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