

# The giant component in a random subgraph of a given graph

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**Abstract.** We consider a random subgraph  $G_p$  of a host graph  $G$  formed by retaining each edge of  $G$  with probability  $p$ . We address the question of determining the critical value  $p$  (as a function of  $G$ ) for which a giant component emerges. Suppose  $G$  satisfies some (mild) conditions depending on its spectral gap and higher moments of its degree sequence. We define the second order average degree  $\tilde{d}$  to be  $\tilde{d} = \sum_v d_v^2 / (\sum_v d_v)$  where  $d_v$  denotes the degree of  $v$ . We prove that for any  $\epsilon > 0$ , if  $p > (1 + \epsilon)/\tilde{d}$  then almost surely the percolated subgraph  $G_p$  has a giant component. In the other direction, if  $p < (1 - \epsilon)/\tilde{d}$  then almost surely the percolated subgraph  $G_p$  contains no giant component.

## 1 Introduction

Almost all information networks that we observe are subgraphs of some host graphs that often have sizes prohibitively large or with incomplete information. A natural question is to deduce the properties that a random subgraph of a given graph must have.

We are interested in random subgraphs of  $G_p$  of a graph  $G$ , obtained as follows: for each edge in  $G_p$  we independently decide to retain the edge with probability  $p$ , and discard the edge with probability  $1 - p$ . A natural special case of this process is the Erdős-Rényi graph model  $G(n, p)$  which is the special case where the host graph is  $K_n$ . Other examples are the percolation problems that have long been studied [10, 11] in theoretical physics, mainly with the host graph being the lattice graph  $\mathbb{Z}^k$ . In this paper, we consider a general host graph, an example of which being a contact graph, consisting of edges formed by pairs of people with possible contact, which is of special interest in the study of the spread of infectious diseases or the identification of community in various social networks.

A fundamental question is to ask for the critical value of  $p$  such that  $G_p$  has a giant connected component, that is a component whose volume is a positive fraction of the total volume of the graph. For the spread of disease on contact networks, the answer to this question corresponds to the problem of finding the epidemic threshold for the disease under consideration, for instance.

For the case of  $K_n$ , Erdős and Rényi answered this in their seminal paper [8]: if  $p = \frac{c}{n}$  for  $c < 1$ , then almost surely  $G$  contains no giant connected component and all components are of size at most  $O(\log n)$ , and if  $c > 1$  then, indeed, there is a giant component of size  $en$ . For general host

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graphs, the answer has been more elusive. Results have been obtained either for very dense graphs or bounded degree graphs. Bollobas, Borgs, Chayes and Riordan [3] showed that for dense graphs (where the degrees are of order  $\Theta(n)$ ), the giant component threshold is  $1/\rho$  where  $\rho$  is the largest eigenvalue of the adjacency matrix. Frieze, Krivelevich and Martin [9] consider the case where the host graph is  $d$ -regular with adjacency eigenvalue  $\lambda$  and they show that the critical probability is close to  $1/d$ , strengthening earlier results on hypercubes [2] and Cayley graphs [12]. For expander graphs with degrees bounded by  $d$ , Alon, Benjamini and Stacey [1] proved that the percolation threshold is greater than or equal to  $1/(2d)$ .

Here, we are interested in percolation on graphs which are not necessarily regular, and can be relatively sparse (i.e.,  $o(n^2)$ .) As we state our results, we note that  $\tilde{d}$  denotes the second-order average degree of  $G$  and  $\sigma$  denotes the spectral gap of the normalized Laplacian of  $G$  and  $\text{vol}_k(G)$  denotes the  $k$ th moment of the degree sequence. Full definitions of these concepts are given in section 2. Further, recall that  $f(n)$  is  $O(g(n))$  if  $\limsup_{n \rightarrow \infty} f(n)/g(n) < \infty$ , and  $f(n)$  is  $o(g(n))$  if  $\lim_{n \rightarrow \infty} f(n)/g(n) = 0$ .

We will prove the following

**Theorem 1.** *Suppose  $G$  has the maximum degree  $\Delta$  satisfying  $\Delta = o(\frac{\tilde{d}}{\sigma})$ . For  $p \leq \frac{1-c}{d}$ , a.a.s. every connected component in  $G_p$  has volume at most  $O(\sqrt{\text{vol}_2(G)}g(n))$ , where  $g(n)$  is any slowly growing function as  $n \rightarrow \infty$ .*

Here, an event occurring a.a.s. indicates that it occurs with probability tending to one as  $n$  tends to infinity. In order to prove the emergence of giant component where  $p \geq (1+c)/\tilde{d}$ , we need to consider some additional conditions. Suppose there is a set  $U$  satisfying

- (i)  $\text{vol}_2(U) \geq (1-\epsilon)\text{vol}_2(G)$ .
- (ii)  $\text{vol}_3(U) \leq Md\text{vol}_2(G)$

where  $\epsilon$  and  $M$  are constant independent of  $n$ . In this case, we say  $G$  is  $(\epsilon, M)$ -admissible and  $U$  is an  $(\epsilon, M)$ -admissible set.

**Theorem 2.** *Suppose  $p \geq \frac{1+c}{\tilde{d}}$  for some  $c \leq \frac{1}{20}$ . Suppose  $G$  satisfies  $\Delta = o(\frac{\tilde{d}}{\sigma})$ ,  $\Delta = o(\frac{d\sqrt{n}}{\log n})$  and  $\sigma = o(n^{-\kappa})$  for some  $\kappa > 0$ , and  $G$  is  $(\frac{c\kappa}{10}, M)$ -admissible. Then a.a.s. there is a unique giant connected component in  $G_p$  with volume  $\theta(\text{vol}(G))$ , and no other component has volume more than  $\max(2d \log n, \omega(\sigma \sqrt{\text{vol}(G)}))$ .*

Here, recall that  $f(n) = \Theta(g(n))$  if  $f(n) = O(g(n))$  and  $g(n) = O(f(n))$ . In this case, we say that  $f$  and  $g$  are of the same order. Also,  $f(n) = \omega(g(n))$  if  $g(n) = o(f(n))$ .

We note that under the assumption that the maximum degree  $\Delta$  of  $G$  satisfying  $\Delta = o(\frac{\tilde{d}}{\sigma})$ , it can be show that the spectral norm of the adjacency matrix satisfies  $\|A\| = \rho = (1 + o(1))\tilde{d}$ .

Further observe in the case that  $\text{vol}_3(G) \leq M d \text{vol}_2(G)$ ,  $G$  is  $(\epsilon, M)$ -admissible for any  $\epsilon$ , and if the other conditions of Theorem 2 are satisfied, we observe that the percolation threshold of  $G$  is  $\frac{1}{d}$ .

The paper is organized as follows: In Section 2 we introduce the notation and some basic facts. In Section 3, we examine several spectral lemmas which allow us to control the expansion. In Section 4, we prove Theorem 1, and in Section 5, we complete the proof of Theorem 2.

## 2 Preliminaries

Suppose  $G$  is a connected graph on vertex set  $V$ . Throughout the paper,  $G_p$  denotes a random subgraph of  $G$  obtained by retaining each edge of  $G$  independently with probability  $p$ .

Let  $A = (a_{uv})$  denote the adjacency matrix of  $G$ , defined by

$$a_{uv} = \begin{cases} 1 & \text{if } \{u, v\} \text{ is an edge;} \\ 0 & \text{otherwise.} \end{cases}$$

We let  $d_v = \sum_u a_{uv}$  denote the degree of vertex  $v$ . Let  $\Delta = \max_v d_v$  denote the maximum degree of  $G$  and  $\delta = \min_v d_v$  denote the minimum degree. For each vertex set  $S$  and a positive integer  $k$ , we define the  $k$ -th volume of  $G$  to be

$$\text{vol}_k(S) = \sum_{v \in S} d_v^k.$$

The volume  $\text{vol}(G)$  is simply the sum of all degrees, i.e.  $\text{vol}(G) = \text{vol}_1(G)$ . We define the average degree  $d = \frac{1}{n} \text{vol}(G) = \frac{\text{vol}_1(G)}{\text{vol}_0(G)}$  and the second order average degree  $\tilde{d} = \frac{\text{vol}_2(G)}{\text{vol}_1(G)}$ .

Let  $D = \text{diag}(d_{v_1}, d_{v_2}, \dots, d_{v_n})$  denote the diagonal degree matrix. Let  $\mathbf{1}$  denote the column vector with all entries 1 and  $\mathbf{d} = D\mathbf{1}$  be column vector of degrees. The normalized Laplacian of  $G$  is defined as

$$\mathcal{L} = I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}}.$$

The spectrum of the Laplacian is the eigenvalues of  $\mathcal{L}$  sorted in increasing order.

$$0 = \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{n-1}.$$

Many properties of  $\lambda_i$ 's can be found in [4]. For example, the least eigenvalue  $\lambda_0$  is always equal to 0. We have  $\lambda_1 > 0$  if  $G$  is connected and  $\lambda_{n-1} \leq 2$  with equality holding only if  $G$  has a bipartite component. Let  $\sigma = \max\{1 - \lambda_1, \lambda_{n-1} - 1\}$ . Then  $\sigma < 1$  if  $G$  is connected and non-bipartite. Furthermore,  $\sigma$  is closely related to the mixing rate of random walks on  $G$ .

The following lemma measures the difference of adjacency eigenvalue and  $\tilde{d}$  using  $\sigma$ .

**Lemma 1.** *The largest eigenvalue of the adjacency matrix of  $G$ ,  $\rho$ , satisfies*

$$|\rho - \tilde{d}| \leq \sigma \Delta.$$

**Proof:** Recall that  $\varphi = \frac{1}{\sqrt{\text{vol}(G)}}D^{1/2}\mathbf{1}$  is the unit eigenvector of  $\mathcal{L}$  corresponding to eigenvalue 0. We have

$$\|I - \mathcal{L} - \varphi\varphi^*\| \leq \sigma.$$

Then,

$$\begin{aligned} |\rho - \tilde{d}| &= \left| \|A\| - \left\| \frac{\mathbf{d}\mathbf{d}^*}{\text{vol}(G)} \right\| \right| \\ &\leq \left\| A - \frac{\mathbf{d}\mathbf{d}^*}{\text{vol}(G)} \right\| \\ &= \|D^{1/2}(I - \mathcal{L} - \varphi\varphi^*)D^{1/2}\| \\ &\leq \|D^{1/2}\| \cdot \|I - \mathcal{L} - \varphi\varphi^*\| \cdot \|D^{1/2}\| \\ &= \sigma\Delta. \end{aligned}$$

□

For any subset of the vertices,  $S$ , we let  $\bar{S}$  denote the complement set of  $S$ . The vertex boundary of  $S$  in  $G$ , denoted by  $\Gamma^G(S)$  is defined as follows:

$$\Gamma^G(S) = \{u \notin S \mid \exists v \in S \text{ such that } \{u, v\} \in E(G)\}.$$

When  $S$  consists of one vertex  $v$ , we simply write  $\Gamma^G(v)$  for  $\Gamma^G(\{v\})$ . We also write  $\Gamma(S) = \Gamma^G(S)$  if there is no confusion.

Similarly, we can define  $\Gamma^{G_p}(S)$  to be the set of neighbors of  $S$  in our percolated subgraph  $G_p$ .

### 3 Several spectral lemmas

We begin by proving two lemmas, first relating expansion in  $G$  to the spectrum of  $G$ , then giving a probabilistic bound on the expansion in  $G_p$ .

**Lemma 2.** *For two disjoint sets  $S$  and  $T$ , we have*

$$\begin{aligned} \left| \sum_{v \in T} d_v |\Gamma(v) \cap S| - \frac{\text{vol}(S)\text{vol}_2(T)}{\text{vol}(G)} \right| &\leq \sigma \sqrt{\text{vol}(S)\text{vol}_3(T)}. \\ \left| \sum_{v \in T} d_v |\Gamma(v) \cap S|^2 - \frac{\text{vol}(S)^2\text{vol}_3(T)}{\text{vol}(G)^2} \right| &\leq \sigma^2 \text{vol}(S) \max_{v \in T} \{d_v^2\} + 2\sigma \frac{\sqrt{\text{vol}(S)^3\text{vol}_5(T)}}{\text{vol}(G)}. \end{aligned}$$

**Proof:** Let  $\mathbf{1}_S$  (or  $\mathbf{1}_T$ ) be the indicative column vector of the set  $S$  (or  $T$ ) respectively. Note

$$\begin{aligned}\sum_{v \in T} d_v |\Gamma(v) \cap S| &= \mathbf{1}_S^* A D \mathbf{1}_T. \\ \text{vol}(S) &= \mathbf{1}_S^* \mathbf{d}. \\ \text{vol}_2(T) &= \mathbf{d}^* D \mathbf{1}_T.\end{aligned}$$

Here  $\mathbf{1}_S^*$  denotes the transpose of  $\mathbf{1}_S$  as a row vector. We have

$$\begin{aligned}& \left| \sum_{v \in T} d_v |\Gamma(v) \cap S| - \frac{\text{vol}(S) \text{vol}_2(T)}{\text{vol}(G)} \right| \\ &= \left| \mathbf{1}_S^* A D \mathbf{1}_T - \frac{1}{\text{vol}(G)} \mathbf{1}_S^* \mathbf{d} \mathbf{d}^* D \mathbf{1}_T \right| \\ &= \left| \mathbf{1}_S^* D^{\frac{1}{2}} (D^{-\frac{1}{2}} A D^{-\frac{1}{2}} - \frac{1}{\text{vol}(G)} D^{\frac{1}{2}} \mathbf{1} \mathbf{1}^* D^{\frac{1}{2}}) D^{\frac{3}{2}} \mathbf{1}_T \right|\end{aligned}$$

Let  $\varphi = \frac{1}{\sqrt{\text{vol}(G)}} D^{1/2} \mathbf{1}$  denote the eigenvector of  $I - \mathcal{L}$  for the eigenvalue 1. The matrix  $I - \mathcal{L} - \varphi \varphi^*$ , which is the projection of  $I - \mathcal{L}$  to the hyperspace  $\varphi^\perp$ , has  $L_2$ -norm  $\sigma$ .

We have

$$\begin{aligned}\left| \sum_{v \in T} d_v |\Gamma(v) \cap S| - \frac{\text{vol}(S) \text{vol}_2(T)}{\text{vol}(G)} \right| &= \left| \mathbf{1}_S^* D^{\frac{1}{2}} (I - \mathcal{L} - \varphi \varphi^*) D^{\frac{3}{2}} \mathbf{1}_T \right| \\ &\leq \sigma \|D^{\frac{1}{2}} \mathbf{1}_S\| \cdot \|D^{\frac{3}{2}} \mathbf{1}_T\| \\ &\leq \sigma \sqrt{\text{vol}(S) \text{vol}_3(T)}.\end{aligned}$$

Let  $e_v$  be the column vector with  $v$ -th coordinate 1 and 0 else where. Then  $|\Gamma^G(v) \cap S| = \mathbf{1}_S^* A e_v$ . We have

$$\sum_{v \in T} d_v |\Gamma^G(v) \cap S|^2 = \sum_{v \in T} d_v \mathbf{1}_S^* A e_v e_v^* A \mathbf{1}_S = \mathbf{1}_S^* A D_T A \mathbf{1}_S.$$

Here  $D_T = \sum_{v \in T} d_v e_v e_v^*$  is the diagonal matrix with degree entry at vertex in  $T$  and 0 else where. We have

$$\begin{aligned}& \left| \sum_{v \in T} d_v |\Gamma^G(v) \cap S|^2 - \frac{\text{vol}(S)^2 \text{vol}_3(T)}{\text{vol}(G)^2} \right| \\ &= \left| \mathbf{1}_S^* A D_T A \mathbf{1}_S - \frac{1}{\text{vol}(G)^2} \mathbf{1}_S^* \mathbf{d} \mathbf{d}^* D_T \mathbf{d} \mathbf{d}^* \mathbf{1}_S \right| \\ &\leq \left| \mathbf{1}_S^* A D_T A \mathbf{1}_S - \frac{1}{\text{vol}(G)} \mathbf{1}_S^* \mathbf{d} \mathbf{d}^* D_T A \mathbf{1}_S \right|\end{aligned}$$

$$\begin{aligned}
& + \left| \frac{1}{\text{vol}(G)} \mathbf{1}_S^* \mathbf{d} \mathbf{d}^* D_T A \mathbf{1}_S - \frac{1}{\text{vol}(G)^2} \mathbf{1}_S^* \mathbf{d} \mathbf{d}^* D_T \mathbf{d} \mathbf{d}^* \mathbf{1}_S \right| \\
& = |\mathbf{1}_S D^{\frac{1}{2}} (I - \mathcal{L} - \varphi \varphi^*) D^{\frac{1}{2}} D_T A \mathbf{1}_S| \\
& \quad + \left| \frac{1}{\text{vol}(G)} \mathbf{1}_S^* \mathbf{d} \mathbf{d}^* D_T D^{\frac{1}{2}} (I - \mathcal{L} - \varphi \varphi^*) D^{\frac{1}{2}} \mathbf{1}_S \right| \\
& \leq |\mathbf{1}_S D^{\frac{1}{2}} (I - \mathcal{L} - \varphi^* \varphi) D^{\frac{1}{2}} D_T D^{\frac{1}{2}} (I - \mathcal{L} - \varphi \varphi^*) D^{\frac{1}{2}} \mathbf{1}_S| \\
& \quad + 2 \left| \frac{1}{\text{vol}(G)} \mathbf{1}_S^* \mathbf{d} \mathbf{d}^* D_T D^{\frac{1}{2}} (I - \mathcal{L} - \varphi \varphi^*) D^{\frac{1}{2}} \mathbf{1}_S \right| \\
& \leq \sigma^2 \text{vol}(S) \max_{v \in T} \{d_v^2\} + 2\sigma \frac{\sqrt{\text{vol}(S)^3 \text{vol}_5(T)}}{\text{vol}(G)}.
\end{aligned}$$

□

**Lemma 3.** *Suppose that two disjoint sets  $S$  and  $T$  satisfy*

$$\text{vol}_2(T) \geq \frac{5p}{2\delta} \sigma^2 \max_{v \in T} \{d_v^2\} \text{vol}(G) \quad (1)$$

$$\frac{25\sigma^2 \text{vol}_3(T) \text{vol}(G)^2}{\delta^2 \text{vol}_2(T)^2} \leq \text{vol}(S) \leq \frac{2\delta \text{vol}_2(T) \text{vol}(G)}{5p \text{vol}_3(T)} \quad (2)$$

$$\text{vol}(S) \leq \frac{\delta^2 \text{vol}_2(T)^2}{25p^2 \sigma^2 \text{vol}_5(T)}. \quad (3)$$

Then we have that

$$\text{vol}(\Gamma^{G_p}(S) \cap T) > (1 - \delta)p \frac{\text{vol}_2(T)}{\text{vol}(G)} \text{vol}(S).$$

with probability at least  $1 - \exp\left(-\frac{\delta(1-\delta)p \text{vol}_2(T) \text{vol}(S)}{10\Delta \text{vol}(G)}\right)$ .

**Proof:** For any  $v \in T$ , let  $X_v$  be the indicative random variable for  $v \in \Gamma^{G_p}(S)$ . We have

$$\mathbb{P}(X_v = 1) = 1 - (1 - p)^{|\Gamma^G(v) \cap S|}.$$

Let  $X = |\Gamma^{G_p}(S) \cap T|$ . Then  $X$  is the sum of independent random variables  $X_v$ .

$$X = \sum_{v \in T} d_v X_v.$$

Note that

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{v \in T} d_v \mathbb{E}(X_v) \\
&= \sum_{v \in T} d_v (1 - (1 - p)^{|\Gamma^G(v) \cap S|})
\end{aligned}$$

$$\begin{aligned}
&\geq \sum_{v \in T} d_v (p|\Gamma^G(v) \cap S| - \frac{p^2}{2} |\Gamma^G(v) \cap S|^2) \\
&= p \sum_{v \in T} d_v |\Gamma^G(v) \cap S| - \frac{p^2}{2} \sum_{v \in T} d_v |\Gamma^G(v) \cap S|^2 \\
&\geq p \left( \frac{\text{vol}(S) \text{vol}_2(T)}{\text{vol}(G)} - \sigma \sqrt{\text{vol}(S) \text{vol}_3(T)} \right) \\
&\quad - \frac{p^2}{2} \left( \frac{\text{vol}(S)^2 \text{vol}_3(T)}{\text{vol}(G)^2} + \sigma^2 \text{vol}(S) \max_{v \in T} \{d_v^2\} + 2\sigma \frac{\sqrt{\text{vol}(S)^3 \text{vol}_5(T)}}{\text{vol}(G)} \right) \\
&> \left(1 - \frac{4}{5}\delta\right) p \frac{\text{vol}_2(T)}{\text{vol}(G)} \text{vol}(S)
\end{aligned}$$

by using Lemma 2 and the assumptions on  $S$  and  $T$ .

We apply the following Chernoff inequality, see e.g. [7]

$$\mathbb{P}(X \leq \mathbb{E}(X) - a) \leq e^{-\frac{a^2}{2 \sum d_v^2 \mathbb{E}(X)^2}} \leq e^{-\frac{a^2}{2 \Delta \mathbb{E}(X)}}.$$

We set  $a = \alpha \mathbb{E}(X)$ , with  $\alpha$  chosen so that  $(1 - \alpha)(1 - \frac{4}{5}\delta) = (1 - \delta)$ . Then

$$\begin{aligned}
\mathbb{P}(X \leq (1 - \delta)p \frac{\text{vol}_2(T)}{\text{vol}(G)} \text{vol}(S)) &< \mathbb{P}(X \leq (1 - \alpha)\mathbb{E}(X)) \\
&\leq \exp\left(-\frac{\alpha^2 \mathbb{E}(X)}{2\Delta}\right) \\
&< \exp\left(-\frac{\alpha(1 - \delta)p \text{vol}_2(T) \text{vol}(S)}{2\Delta \text{vol}(G)}\right).
\end{aligned}$$

To complete the proof, note  $\alpha > \delta/5$ . □

## 4 The range of $p$ with no giant component

In this section, we will prove Theorem 1.

**Proof of Theorem 1:** It suffices to prove the following claim.

**Claim A:** If  $p\rho < 1$ , where  $\rho$  is the largest eigenvalue of the adjacency matrix, with probability at least  $1 - \frac{1}{C^2(1-p\rho)}$ , all components have volume at most  $C\sqrt{\text{vol}_2(G)}$ .

*Proof of Claim A:* Let  $x$  be the probability that there is a component of  $G_p$  having volume greater than  $C\sqrt{\text{vol}_2(G)}$ . Now we choose two random vertices with the probability of being chosen proportional to their degrees in  $G$ . Under the condition that there is a component with volume greater

than  $C\sqrt{\text{vol}_2(G)}$ , the probability of each vertex in this component is at least  $\frac{C\sqrt{\text{vol}_2(G)}}{\text{vol}(G)}$ . Therefore, the probability that the random pair of vertices are in the same component is at least

$$x \left( \frac{C\sqrt{\text{vol}_2(G)}}{\text{vol}(G)} \right)^2 = \frac{C^2 x \tilde{d}}{\text{vol}(G)}. \quad (4)$$

On the other hand, for any fixed pair of vertices  $u$  and  $v$  and any path  $P$  of length  $k$  in  $G$ . The probability of  $u$  and  $v$  is connected by this path in  $G_p$  is exactly  $p^k$ . The number of  $k$ -path from  $u$  to  $v$  is at most  $\mathbf{1}_u^* A^k \mathbf{1}_v$ . Since the probabilities of  $u$  and  $v$  being selected are  $\frac{d_u}{\text{vol}(G)}$  and  $\frac{d_v}{\text{vol}(G)}$  respectively, the probability that the random pair of vertices are in the same connected component is at most

$$\sum_{u,v} \frac{d_u}{\text{vol}(G)} \frac{d_v}{\text{vol}(G)} \sum_{k=0}^n p^k \mathbf{1}_u^* A^k \mathbf{1}_v = \sum_{k=0}^n \frac{1}{\text{vol}(G)^2} p^k \mathbf{d}^* A^k \mathbf{d}.$$

We have

$$\begin{aligned} \sum_{k=0}^n \frac{1}{\text{vol}(G)^2} p^k \mathbf{d}^* A^k \mathbf{d} &\leq \sum_{k=0}^{\infty} \frac{p^k \rho^k \text{vol}_2(G)}{\text{vol}(G)^2} \\ &\leq \frac{\tilde{d}}{(1-p\rho)\text{vol}(G)}. \end{aligned}$$

Combining with (4), we have

$$\frac{C^2 x \tilde{d}}{\text{vol}(G)} \leq \frac{\tilde{d}}{(1-p\rho)\text{vol}(G)},$$

which implies

$$x \leq \frac{1}{C^2(1-p\rho)}.$$

Claim A is proved. □

## 5 The emergence of the giant component

**Lemma 4.** *Suppose  $G$  contains an  $(\epsilon, M)$ -admissible set  $U$ . Then we have*

1.  $\tilde{d} \leq \frac{M}{(1-\epsilon)^2} d$ .
2. For any  $U' \subset U$  with  $\text{vol}_2(U') > \eta \text{vol}_2(U)$ , we have

$$\text{vol}(U') \geq \frac{\eta^2(1-\epsilon)\tilde{d}}{Md} \text{vol}(G).$$

**Proof:** Since  $G$  is  $(\epsilon, M)$ -admissible, we have a set  $U$  satisfying

- (i)  $\text{vol}_2(U) \geq (1 - \epsilon)\text{vol}_2(G)$
- (ii)  $\text{vol}_3(U) \leq Md\text{vol}_2(G)$ .

We have

$$\tilde{d} = \frac{\text{vol}_2(G)}{\text{vol}(G)} \leq \frac{\text{vol}_2(G)}{\text{vol}(U)} \leq \frac{1}{1 - \epsilon} \frac{\text{vol}_2(U)}{\text{vol}(U)} \leq \frac{1}{1 - \epsilon} \frac{\text{vol}_3(U)}{\text{vol}_2(U)} \leq \frac{M}{(1 - \epsilon)^2} d.$$

For any  $U' \subset U$  with  $\text{vol}_2(U') > \eta\text{vol}_2(U)$ , we have

$$\text{vol}(U') \geq \frac{\text{vol}_2(U')^2}{\text{vol}_3(U')} \geq \frac{\eta^2 \text{vol}_2(U)^2}{\text{vol}_3(U)} \geq \frac{\eta^2 (1 - \epsilon) \text{vol}_2(G)}{Md} \geq \frac{\eta^2 (1 - \epsilon) \tilde{d}}{Md} \text{vol}(G).$$

□

**Proof of Theorem 2:** It suffices to assume  $p = \frac{1+c}{d}$  for some  $c < \frac{1}{20}$ .

Let  $\epsilon = \frac{c\kappa}{10}$  be a small constant, and  $U$  be a  $(\epsilon, M)$ -admissible set in  $G$ . Define  $U'$  to be the subset of  $U$  containing all vertices with degree at least  $\sqrt{\epsilon}d$ . We have

$$\begin{aligned} \text{vol}_2(U') &\geq \text{vol}_2(U) - \sum_{d_v < \sqrt{\epsilon}d} d_v^2 \\ &\geq (1 - \epsilon)\text{vol}_2(G) - \epsilon nd^2 \\ &\geq (1 - 2\epsilon)\text{vol}_2(G). \end{aligned}$$

Hence,  $U'$  is a  $(2\epsilon, M)$  admissible set. We will concentrate on the neighborhood expansion within  $U'$ .

Let  $\delta = \frac{c}{2}$  and  $C = \frac{25M}{\delta^2(1-4\epsilon)^2}$ . Take an initial set  $S_0 \subset U'$  with  $\max(C\sigma^2\text{vol}(G), \Delta \ln n) \leq \text{vol}(S_0) \leq \max(C\sigma^2\text{vol}(G), \Delta \ln n) + \Delta$ .

Let  $T_0 = U' \setminus S_0$ . For  $i \geq 1$ , we will recursively define  $S_i = \Gamma^{G_p}(S_{i-1}) \cap U'$  and  $T_i = U' \setminus \cup_{j=0}^i S_j$  until  $\text{vol}_2(T_i) \leq (1 - 3\epsilon)\text{vol}_2(G)$  or  $\text{vol}(S_i) \geq \frac{2\delta\text{vol}_2(T_i)\text{vol}(G)}{5p\text{vol}_3(T_i)}$ .

Condition 1 in Lemma 3 is always satisfied.

$$\begin{aligned} \frac{5p}{2\delta} \sigma^2 \max_{v \in T_i} d_v^2 \text{vol}(G) &\leq \frac{5(1+c)}{2\tilde{d}\delta} \sigma^2 \Delta^2 \text{vol}(G) \\ &= \left(\frac{\sigma\Delta}{\tilde{d}}\right)^2 \frac{5(1+c)}{2\delta} \text{vol}_2(G) \\ &= o(\text{vol}_2(G)) \\ &\leq \text{vol}_2(T_i). \end{aligned}$$

Condition 3 in Lemma 3 is also trivial because

$$\begin{aligned}
\frac{\delta^2 \text{vol}_2(T_i)^2}{25p^2 \sigma^2 \text{vol}_5(T_i)} &\geq \frac{\delta^2 \text{vol}_2(T_i)^2}{25p^2 \sigma^2 \Delta^2 \text{vol}_3(T_i)} \\
&\geq \frac{\delta^2 (1 - 3\epsilon) \text{vol}_2(G)}{25p^2 \sigma^2 \Delta^2 M d} \\
&\geq \left(\frac{\tilde{d}}{\sigma \Delta}\right)^2 \frac{\delta^2 (1 - 3\epsilon)}{25(1 + c)^2 M} \text{vol}(G) \\
&= \omega(\text{vol}(G)).
\end{aligned}$$

Now we verify condition 2. We have

$$\text{vol}(S_0) > C \sigma^2 \text{vol}(G) = \frac{25M}{\delta^2 (1 - 4\epsilon)^2} \sigma^2 \text{vol}(G) \geq \frac{25\sigma^2 \text{vol}_3(T_0) \text{vol}(G)^2}{\delta^2 \text{vol}_2(T_0)^2}.$$

The conditions of Lemma 3 are all satisfied. Then we have that

$$\text{vol}(\Gamma^{G_p}(S_0) \cap T_0) > (1 - \delta)p \frac{\text{vol}_2(T_0)}{\text{vol}(G)} \text{vol}(S_0).$$

with probability at least  $1 - \exp\left(-\frac{\delta(1-\delta)p \text{vol}_2(T_0) \text{vol}(S_0)}{10\Delta \text{vol}(G)}\right)$ .

Since  $(1 - \delta)p \frac{\text{vol}_2(T_i)}{\text{vol}(G)} \geq (1 - \delta)(1 - 3\epsilon)(1 + c) = \beta > 1$  by our assumption that  $c$  is small, the neighborhood of  $S_i$  grows exponentially, allowing condition 2 of Lemma 3 to continue to hold and us to continue the process. We stop when one of the following two events happens,

- $\text{vol}(S_i) \geq \frac{2\delta \text{vol}_2(T_i) \text{vol}(G)}{5p \text{vol}_3(T_i)}$ .
- $\text{vol}_2(T_i) \leq (1 - 3\epsilon) \text{vol}_2(G)$ .

Let us denote the time that this happens by  $t$ .

If the first, but not the second, case occurs we have

$$\text{vol}(S_t) \geq \frac{2\delta \text{vol}_2(T_t) \text{vol}(G)}{5p \text{vol}_3(T_t)} \geq \frac{2\delta(1 - 3\epsilon)}{5M(1 + c)} \text{vol}(G).$$

In the second case, we have

$$\text{vol}_2\left(\bigcup_{j=0}^t S_j\right) = \text{vol}_2(U') - \text{vol}_2(T_t) \geq \epsilon \text{vol}_2(G) \geq \epsilon \text{vol}(U').$$

By lemma 4 with  $\eta = \epsilon$ , we have  $\text{vol}(\bigcup_{j=0}^t S_j) \geq \frac{\epsilon^2(1-2\epsilon)\tilde{d}}{Md} \text{vol}(G)$ . On the other hand, note that that since  $\text{vol}(S_i) \geq \beta \text{vol}(S_{i-1})$ , we have that  $\text{vol}(S_i) \leq \beta^{i-t} \text{vol}(S_t)$ , and hence we have

$$\text{vol}\left(\bigcup_{j=0}^t S_j\right) \leq \sum_{j=0}^t \beta^{-j} \text{vol}(S_t)$$

so

$$\text{vol}(S_t) \geq \frac{\epsilon^2(1-2\epsilon)\tilde{d}(\beta-1)}{Md\beta}\text{vol}(G).$$

In either case we have  $\text{vol}(S_t) = \Theta(\text{vol}(G))$ . For the moment, we restrict ourselves to the case where  $C\sigma^2n > \Delta \ln n$ .

Each vertex in  $S_t$  is in the same component as some vertex in  $S_0$ , which has size at most  $\frac{\text{vol}(S_0)}{\sqrt{\epsilon d}} \leq C'\sigma^2n$ . We now combine the  $k_1$  largest components to form a set  $W^{(1)}$  with  $\text{vol}(W^{(1)}) > C\sigma^2\text{vol}(G)$ , such that  $k_2$  is minimal. If  $k_1 \geq 2$ ,  $\text{vol}(W^{(1)}) \leq 2C\sigma^2\text{vol}(G)$ . Note that since the average size of a component is  $\frac{\text{vol}(S_t)}{|S_0|} \geq C_1\frac{\text{vol}(G)}{\sigma^2n}$ ,  $k_1 \leq C'_1\sigma^4n$ .

We grow as before: Let  $W_0^{(1)} = W^{(1)}$ ,  $Q_0^{(1)} = T_{t-1} \setminus W_0^{(1)}$ . Note that the conditions for Lemma 3 are satisfied by  $W_0^{(1)}$  and  $T_0^{(1)}$ . We run the process as before, setting  $W_t^{(1)} = \Gamma(W_t^{(1)}) \cap Q_{t-1}^{(1)}$  and  $Q_t^{(1)} = Q_{t-1}^{(1)} \setminus W_t^{(1)}$  stopping when either  $\text{vol}(Q_t^{(1)}) < (1-4\epsilon)\text{vol}_2(G)$  or  $\text{vol}(W_t^{(1)}) > \frac{2\delta\text{vol}_2(Q_t^{(1)})\text{vol}(G)}{5p\text{vol}_3(Q_t^{(1)})} \geq \frac{2\delta(1-4\epsilon)}{5M(1+c)}\text{vol}(G)$ . As before, in either case  $\text{vol}(W_t^{(1)}) = \Theta(\text{vol}(G))$ . Note that if  $k_1 = 1$ , we are now done as all vertices in  $W_t^{(1)}$  lie in the same component of  $G_p$ .

Now we iterate. Each of the vertices in  $W_t^{(1)}$  lies in one of the  $k_1$  components of  $W_0^{(1)}$ . We combine the largest  $k_2$  components to form a set  $W^{(2)}$  of size  $> C\sigma^2\text{vol}(G)$ . If  $k_2 = 1$ , then one more growth finishes us, otherwise  $\text{vol}(W^{(2)}) < 2C\sigma^2\text{vol}(G)$ , the average size of components is at least  $C_2\frac{\text{vol}(G)}{\sigma^4n}$  and hence  $k_2 \leq C'_2\sigma^6n$ .

We iterate, growing  $W^{(m)}$  until either  $\text{vol}(Q_t^{(m)}) < (1-(m+3)\epsilon)\text{vol}_2(G)$  or  $\text{vol}(W_t^{(m)}) > \frac{2\delta\text{vol}_2(Q_t^{(m)})\text{vol}(G)}{5p\text{vol}_3(Q_t^{(m)})}$ , so that  $W_t^{(m)}$  has volume  $\theta(\text{vol}(G))$  and then creating  $W^{(m+1)}$  by combining the largest  $k_{m+1}$  components to form a  $W^{(m+1)}$  with volume at least  $C\sigma^2n$ . Once  $k_m = 1$  for some  $m$  all vertices in  $W^{(m)}$  are in the same component and one more growth round finishes the process, resulting in a giant component in  $G$ . Note that the average size of a component in  $W_n^{(m)}$  has size at least  $\frac{\text{vol}(G)}{\sigma^{2(m+1)}n}$  (that is, components must grow by a factor of at least  $\frac{1}{\sigma^2}$  each iteration) and if  $k_m > 1$ , we must have  $k_m \leq C_m\sigma^{2(m+1)}n$ . If  $m = \lceil \frac{1}{2\kappa} \rceil - 1$ , this would imply that  $k_m = o(1)$  by our condition  $\sigma = o(n^{-\kappa})$ , so after at most  $\lceil \frac{1}{2\kappa} \rceil - 1$  rounds, we must have  $k_m = 1$  and the process will halt with a giant connected component.

In the case where  $\Delta \ln n > C\sigma^2n$ , we note that  $|S_0| \leq \frac{\text{vol}(S_0)}{\sqrt{\epsilon d}} \leq C'\frac{\Delta \ln n}{\sqrt{\epsilon d}}$ , and the average volume of components in  $S_t$  is at least  $\frac{C''\text{vol}(G)d}{\Delta \ln n} = \omega(\Delta \ln n)$ , so we can form  $W^{(1)}$  by taking just 1 component for  $n$  large enough, and the proof goes as above.

We note that throughout, we never try to expand if

$$\text{vol}(Q_t^{(m)}) < (1-(m+3)\epsilon)\text{vol}_2(G) < \left(1 - \left(\frac{1}{2\kappa} + 4\right)\epsilon\right)\text{vol}_2(G) < \left(1 - \frac{9c}{20}\right)\text{vol}_2(G).$$

By our choice of  $c$  being sufficiently small,  $(1 - (m + 3)\epsilon)(1 - \delta)(1 + c) > 1$  at all times, so throughout, noting that  $\text{vol}(S_i)$  and  $\text{vol}(W_i^{(m)})$  are at least  $\Delta \ln n$ , we are guaranteed our exponential growth by Lemma 3 with an error probability bounded by

$$\exp\left(-\frac{\delta(1 - \delta)p\text{vol}_2(T_i)\text{vol}(S_i)}{10\Delta\text{vol}(G)}\right) \leq \exp\left(-\frac{\delta(1 - \delta)(1 - \frac{9c}{20})\text{vol}(S_i)}{2\Delta}\right) \leq n^{-K}$$

We run for a constant number of phases, and run for at most a logarithmic number of steps in each growth phase as the sets grow exponentially. Thus, the probability of failure is at most  $C'' \log(n)n^{-K} = o(1)$  for some constant  $C''$ , thus completing our argument that  $G_p$  contains a giant component with high probability.

Finally, we prove the uniqueness assertion. With probability  $1 - C'' \log(n)n^{-K}$  there is a giant component  $X$ . Let  $u$  be chosen at random; we estimate the probability that  $u$  is in a component of volume at least  $\max(2d \log n, \omega(\sigma \sqrt{\text{vol}(G)}))$ . Let  $Y$  be the component of  $u$ . Theorem 5.1 of [4] asserts that if  $\text{vol}(Y) \geq \max(2d \log n, \omega(\sigma \sqrt{\text{vol}(G)}))$ :

$$e(X, Y) \geq \frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(G)} - \sigma \sqrt{\text{vol}(X)\text{vol}(Y)} \geq 1.5d \log n$$

Note that the probability that  $Y$  is not connected to  $X$  given that  $\text{vol}(Y) = \omega(\sigma \sqrt{\text{vol}(G)})$  is  $(1 - p)^{e(X, Y)} = o(n^{-1})$ , so with probability  $1 - o(1)$  no vertices are in such a component - proving the uniqueness of large components.

□

## References

1. N. Alon, I. Benjamini, and A. Stacey, Percolation on finite graphs and isoperimetric inequalities, *Annals of Probability*, **32**, no. 3 (2004), 1727–1745.
2. M. Ajtai, J. Komlós and E. Szemerédi, Largest random component of a  $k$ -cube, *Combinatorica* **2** (1982), 1–7.
3. B. Bollobas, C. Borgs, J. Chayes, O. Riordan, Percolation on dense graph sequences, preprint.
4. F. Chung, *Spectral Graph Theory*, AMS Publications, 1997.
5. F. Chung, L. Lu, and V. Vu, The spectra of random graphs with given expected degrees, *Internet Mathematics* **1** (2004) 257–275.
6. F. Chung, and L. Lu, Connected components in random graphs with given expected degree sequences, *Annals of Combinatorics* **6**, (2002), 125–145,
7. F. Chung, and L. Lu, *Complex Graphs and Networks*, AMS Publications, 2006.
8. P. Erdős, and A. Rényi, On Random Graphs I, *Publ. Math Debrecen* **6**, (1959), 290–297.
9. A. Frize, M. Krivelevich, R. Martin, The emergence of a giant component of pseudo-random graphs, *Random Structures and Algorithms* **24**, (2004), 42–50.
10. G. Grimmett, *Percolation*, Springer, New York, 1989.
11. H. Kesten, Asymptotics in high dimension for percolation, In *Disorder in Physical Systems* (G. R. Grimmett and D. J. A. Welsh, eds.) Oxford Univ. Press, (1990) 219–240.
12. C. Malon and I. Pak, Percolation on finite Cayley graphs, *Lecture Notes in Comput. Sci.* **2483**, springer, Berlin, (2002), 91–104.