Curves in space. Local theory.

Euclidean space

\( \mathbb{E}^3 \) - affine space with Euclidean dot product of vectors \( \vec{v}, \vec{w} \)
- length \( \| \vec{v} \| = \sqrt{\vec{v} \cdot \vec{v}} \)

Frame: Choice of origin 0 and an orthonormal basis \( \hat{e}_1, \hat{e}_2, \hat{e}_3 \).

Any point \( X \) is determined by three coordinates:

\[
\overrightarrow{OX} = \hat{e}_1 x^1 + \hat{e}_2 x^2 + \hat{e}_3 x^3 = [\hat{e}_1 \hat{e}_2 \hat{e}_3] \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}
\]

Customary to write

\[
X = \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}
\]
- position vector

- unit arrows on vectors
- use \( x^1, x^2, x^3 \) when convenient

A second vector on a vector at a point \( P \) is a pair \((\vec{P}, \vec{V})\). A free vector can be "moved" anywhere in \( \mathbb{E}^3 \).

This is special to Euclidean space.

Parameterized curves

A parameterized curve is a surjective function

\[
X : I \rightarrow \mathbb{E}^3
\]

I - (open) interval in real \( \mathbb{R} \)
\( t \) - parameter

Write

\[
X = \begin{bmatrix} x^1(t) \\ x^2(t) \\ x^3(t) \end{bmatrix}
\]

Smooth: \( x^1(t), x^2(t), x^3(t) \) and \( C^\infty \)
- usually \( C^2 \) or \( C^3 \) suffice
Velocity vector
\[ X = x(t) \text{ curve} \]
\[ V = \frac{dx}{dt} = a(t) \]
"velocity" = a bounded vector
Dot product \( \frac{dt}{dx} \).
\[ V \text{ is a vector field along } x. \]
Regular curve
- \( x(t) \) is class \( C^2 \) or \( C^3 \)
- \( \dot{x}(t) \) is never the zero vector
Note! If \( \dot{x}(t) \) is never zero, then \( x(t) \) is injective in the small.
For \( x(t) = x(t_0) + \dot{x}(x_0)(t - t_0) + O((t-t_0)^2) \)
\[ x(t) - x(t_0) = (\dot{x}(t_0) + O(t-t_0)) (t-t_0) \]
For \( t \nearrow t_0 \), \( x(t) \not= x_0 \) and have \( \dot{x}(t) \not= \dot{x}(t_0) \).

Example. Folium of Descartes
\[ x(t) = \begin{bmatrix} t^2 - 1 \\ t^3 - t \\ 0 \end{bmatrix} \quad t = x^2 + x^3 \]
\[ \dot{x}(t) = \begin{bmatrix} 2t \\ 3t^2 - 1 \\ 0 \end{bmatrix} \text{ never zero.} \]
Injective in the small, but not in the large, \( x(-1) = x(1) \).

Change of parameters
\[ X = x(t) \text{ curve} \]
\[ t = \varphi(x') \text{ \( \varphi \) monotone and } C^2 \text{ everywhere} \]
\[ \frac{dt}{dx'} \text{ never zero} \]
\[ X = x(\varphi(x')) = (x \circ \varphi)(x') \text{ is same curve with new parameter } x'. \]
Write \( X = x(x') \). This is a different \( x. \)
Arc length

Curve \( X = x(t) \) - regular
\[ a < t_1 < t_2 < \cdots < t_n = b \]

Length of polygonal approximation
\[ L = \sum_{i=1}^{n} \| x(t_i) - x(t_{i-1}) \| . \]

Length of curve \( (\text{from } x(a) \text{ to } x(b)) = \operatorname{sup} \{ \text{lengths of polygonal approximations} \} \)
- sup is over partitions \( \{ t_i \} \).

From calculus:

\[ \text{Length} = \int_{a}^{b} \| x'(t) \| \, dt . \]

Note: length does not depend on parameterization - see geometric definition. Can, change parameter

\[ \int_{a}^{b} \left\| \frac{dx}{dt}(t) \right\| \, dt = \int_{a}^{b} \left\| \frac{dx}{dt}(\varphi(t')) \right\| \frac{d\varphi}{dt}, (t') \, dt' \]
\[ = \pm \int_{a}^{b} \left\| \frac{dx}{dt}(\varphi(t')) \right\| \frac{d\varphi}{dt}, (t') \, dt' \]
\[ = \pm \int_{a}^{b} \left\| \frac{d}{dt}(\varphi(t')) \right\| \, dt' \]

Arc length on parameter

\( X = x(t) \) \( a \leq t \leq b \) - regular curve

\[ s = \int_{a}^{b} \| x'(t) \| \, dt \] arc length along \( \chi \) from \( \varphi(a) \) to \( \varphi(b) \)

\( \frac{ds}{dt} = \| x'(t) \| \) is never zero.

\( t = \varphi^{1}(s) \) - value for \( t \) as function of \( s \)

\( X = x(\varphi(s)) \) \( 0 \leq s \leq L \) \( L \) = length of \( X \)

on \( \varphi(t) \)
Problem 1.1 Compute the arc lengths of:

a) the line \( y = mx + b \), \( 0 \leq x \leq 6 \)

b) the circle \( x = a \cos t \), \( 0 \leq x \leq 2\pi \)

\[ y = a \sin t \]

c) the logarithmic spiral (in polar coordinates) \( r = a e^{-k\theta} \), \( 0 \leq \theta \leq \infty \)

d) the astroid \( x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \)

Hint: \( x = a \cos^3 t \), \( y = a \sin^3 t \).

Parameterize these curves by their arc length as much as possible.

The moving frame

Unit tangent vector

\[ \frac{\alpha'(t)}{\|\alpha'(t)\|} \]

Unit normal vector

\[ \lim_{{h \to 0}} \frac{\alpha(t + h) - \alpha(t)}{\|\alpha(t + h) - \alpha(t)\|} \]

T is in the direction of increasing \( t \) on an arc length.

Note: \( \frac{d\alpha}{dt} = T \cdot \text{unit normal} \)

If \( f(t) \) is any function, \( \frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} \).

So

\[ T = \frac{\frac{d\alpha}{dt}}{\| \frac{d\alpha}{dt} \|} = \frac{\frac{dx}{dt}}{\| \frac{dx}{dt} \|} = \frac{\frac{d\alpha}{dt}}{\| \frac{d\alpha}{dt} \|} \]

If \( x = \alpha(t) \) is parameterized by arc length, then \( T = \alpha'(t) \), \( \| T \| = 1 \), \( \frac{d}{dt} = \frac{d\alpha}{dt} \).

Principal normal

The principal normal is based at \( \alpha(t) \).

\[ \frac{T'(t)}{\| T'(t) \|} \]

The principal normal is based at \( \alpha(t) \).

Note: \( T(t) \cdot T'(t) = 0 \).
Osculating plane

\[ B(\hat{t}_1, \hat{t}_2, \hat{t}_3), X - \phi(\hat{t}_1, \hat{t}_2, \hat{t}_3) = 0 \]

**Unit vector equation of plane**

\[ \hat{N}(\hat{t}_1, \hat{t}_2, \hat{t}_3) \]

**Not collinear**

\[ B \text{ is a unit vector} \]

**A normal to the osculating plane is a unit vector.**

\[ x'(t_0) \cdot \hat{N}(t_0) = 0 \]

\[ B(t_0) = \text{line } B(\hat{t}_1, \hat{t}_2, \hat{t}_3) \]

\[ B(t_0), x'(t_0) = 0 \text{ and } B(t_0) \cdot x''(t_0) = 0 \]

So, \( B(t) \) (any \( t \)) is a unit vector parallel to \( x'(t) \times x''(t) \). Need \( \alpha' \times \alpha'' \)

\[ \alpha'(t) \cdot \alpha'(t) = 1 \text{ gives } x'(t) \times x''(t) = 0 \]

\[ \alpha', \alpha'', \alpha' \times \alpha'' \] are mutually orthogonal.

The unit vector \( B(t) \) is the binormal to the curve \( x \) at \( x(t) \), a bounded vector.

Osculating circle

\[ \alpha(t) \text{ regular curve} \]

\[ \alpha(t), \alpha'(t_0), \alpha'(t_1), \alpha'(t_2), \alpha'(t_3) \text{ not collinear} \]

\[ \alpha(t_1), \alpha(t_2), \alpha(t_3) \text{ determine a circle in the plane } B(\hat{t}_1, \hat{t}_2, \hat{t}_3), X - \phi(\hat{t}_1, \hat{t}_2, \hat{t}_3) = 0 \]

\[ C(t_1, t_2, t_3) \text{ center of circle} \]

\[ R(t_1, t_2, t_3) \text{ radius} \]

\[ R(t_1, t_2, t_3) = \text{real number} \]
Equation of circle in plane above:

\[(X - C(1, 2, 3)) \cdot (X - C(1, 2, 3)) = (R(1, 2, 3))^2\]

The function

\[\Phi := (X(1) - C(1, 2, 3)) \cdot (X(1) - C(1, 2, 3)) - (R(1, 2, 3))^2\]

has zero at \(1, 2, 3\).

Problem 1.2 Use Rolle's Theorem three times, as before. Take the limit as \(t_1, t_2, t_3 \to 0\) and obtain

for \(C(1) = \lim C(1, t_2, t_3)\)

\[R(1) = \lim R(1, t_2, t_3)\]

and hence for any \(t\),

1. \(B(t) \cdot (X(t) - C(t)) = 0\)
2. \((X(t) - C(t)) \cdot (X(t) - C(t)) = (R(t))^2\)
3. \((X(t) - C(t)) \cdot X''(t) = 0\)
4. \((X(t) - C(t)) \cdot X''(t) + 1 = 0\)

Consequence of Prob 1.2:

1. \(X(t) - C(t)\) is perpendicular to \(B(t)\)
2. \(X(t) - C(t)\) is parallel to \(R(t)\)

By (4):

\[\lambda^2 = (R(t))^2, \quad \lambda = -R(t)\]

\[X(t) = X(t) + \lambda(t) R(t)\]

- center of curvature
- radius of curvature
- at \(X(t)\)

Note, \(X''(t)\) is a multiple of \(N(t)\):

\[X''(t) = \lambda(t) X(t) + R(t)\]

Then (3) says:

\[-X(t) R(t) + \lambda(t) X(t) + 1 = 0\]

so

\[X(t) = \frac{1}{R(t)}, \quad \text{curvature of} \ X \ at \ X(t)\]

Note: \(X''(t) = T(t)\), 

\[\frac{dT}{dt} = X(t), \quad t = \parallel T \parallel\]
Remark: \( T(t) \) is identically zero iff \( \mathbf{x} \) is a straight line. For all \( t \), mean \( T'(t) = 0 \) or \( x''(t) = 0 \) for all \( t \). Also \( x(t) = x(0) + Tt \), with \( T \) a constant vector.

**Torsion**

\[
\frac{d}{dt} (T \times N) = \frac{T' \times x' + T \times N'}{x'} = 0 \quad \text{by} \quad T' = N \times x'
\]

So \( \frac{d}{dt} \) is perpendicular to \( T \). And, \( B \cdot B = 1 \), \( B' \cdot B = 0 \), so \( \frac{d}{dt} \) is perpendicular to \( B \). Then, \( \frac{d}{dt} \) is parallel to \( N \):

\[
\frac{d}{dt} \mathbf{B} = -N \mathbf{T}, \quad T(t) = \|B(t)\|, \quad T(t) \text{ is the torsion of } \mathbf{x} \text{ at } x(t).
\]

Remark: \( T(t) \) is identically zero iff \( \mathbf{x} \) is a plane curve. For all \( t \), mean \( \frac{d}{dt} = 0 \), so \( B = \text{constant vector} \). Then, \( \frac{d}{dt} (B \times x' \| = B \cdot x' = B \cdot T = 0 \), so \( B \times x' \| = \text{constant} \). The concept is trivial.

Note the functions \( x(t) \) and \( T(t) \) are invariance under rigid motions. Congruent curves have the same curvature and torsion. Later, the content will be proved.
Problem 3 Let \( \alpha \) be a plane curve. If \( \theta(t) \) is the angle between the tangent vector \( \alpha'(t) \) at \( x(t) \) and a fixed direction in the plane, show that \( x(t) = \frac{d\theta}{dt} u \). Note: Plane curves can have signed curvature, curves in space do not.

Hint: \( T = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \) and \( \frac{dT}{dt} = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} \frac{d\theta}{dt} \).

Example The ellipse \( \alpha(t) = \begin{bmatrix} a \cos t \\ b \sin t \\ 0 \end{bmatrix} \).

Use \( \frac{d\alpha}{dt} = \frac{dx}{dt} \frac{dt}{dt} \).

\( \alpha = Ti = \begin{bmatrix} b \cos t \\ b \sin t \\ 0 \end{bmatrix} \), \( T = \frac{d\alpha}{dt} = \begin{bmatrix} -a \sin t \\ b \cos t \end{bmatrix} \), unit vector,

\( \dot{T} = a^2 \sin^2 \theta + b^2 \cos^2 \theta \).

Then \( \ddot{T} = (\ddot{T}i)i + \ddot{T} \dot{T} = \ddot{T}i + \dot{T}N \times i \).

Vector product \( \ddot{T} = \text{ACCELERATION} \).

\( \dddot{T}i = \ddot{T} \times (\dddot{T}i + N \times \dddot{T}i) = \dot{B} \times i \).

\( \dot{B} = \begin{bmatrix} -a \cos t \\ b \sin t \\ 0 \end{bmatrix} \times \begin{bmatrix} -a \cos t \\ b \sin t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ ab \end{bmatrix} \).

Thus \( \dot{B} \times i = ab \). Thus

\( K = \frac{ab}{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}} \).

Remark The same technique can be used for curves in space.

Example The twisted curve

\( x(t) = \begin{bmatrix} x \\ z^2 \\ t^3 \end{bmatrix} \).
(1) \[ T' = \dot{x} = \begin{bmatrix} 1 \\ \frac{d}{dx} \end{bmatrix}, \quad T^2 = 1 + 4x^2 + 9x^4 \]

(2) \[(T'^2 + T^2) = Nx'^2 + T' \\ = \begin{bmatrix} 0 \\ 6x \end{bmatrix}\]

Vector product (11 \times 12):

\[ T \times (Nx'^2 + T') = \begin{bmatrix} 1 \\ 2x \end{bmatrix} \times \begin{bmatrix} 2t \\ 2 \\ 0 \end{bmatrix} \]

(3) \[ B \times i^3 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \]

Differentiate (3):

\[ \frac{d}{dx} (B \times i^3) = B \frac{d}{dx} (x \times i^3) = \begin{bmatrix} 6x^2 \\ -6x \\ 0 \end{bmatrix} \]

Scalar product (12) \times (4):

\[ -Nx'^2 = \frac{d}{dx} (x \times i^3) = \begin{bmatrix} 12x \\ -6x \\ 0 \end{bmatrix} \]

Scalar product (3) \times (3) left-handed.

(5) \[ -x^2 \dot{x} + 6 = -12 \]

Scalar product (3) \times (3)

(6) \[ \dot{x}^2 = 36x^4 + 36x^2 + 4 \]

Solve (1), (6), (5) for \( \dot{x}^2 = \frac{4(1 + 9x^2 + 9x^4)}{(1 + 4x^2 + 9x^4)^3}, \quad \tau = \frac{3}{1 + 4x^2 + 9x^4} \]

Problem (1.4) let \( x \) be a regular curve parameterized by \( t \). """denotes """Show:

\[ \dot{x} = \frac{11 \dot{x} \times \ddot{x}}{11 \ddot{x} \times \ddot{x}}, \quad \tau = \frac{\ddot{x} \times \ddot{x} \times \ddot{x}}{11 \ddot{x} \times \ddot{x} \times \ddot{x}}. \]

Remark. One can compute the curvature and torsion of curves defined implicitly as the intersection of two surfaces \( f(x, y, z) = 0 \) and \( g(x, y, z) = 0 \). It begins by noting that the tangent vector \( T \) at a point is parallel to \( \nabla f \times \nabla g \). (\( \nabla \) denotes gradient.) See Willmore p. 16 ff.
Problem 15

Investigate the helix \( x \) on a right circular cone given by

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
\cos \omega t \\
\sin \omega t \\
0
\end{bmatrix} (\sin a) t + \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} (\cos a) t,
\]

where \( a \) is the vertex angle of the cone.

a) Find the curvature and torsion.

b) Reconsider the "limiting cases" of \( a = 0 \) (helix on a circular cylinder) and \( a = 90^\circ \) (planar spiral).

Problem 16

For the curve \( X(t) = (a + a \cos t, a \sin t, a \sin \frac{t}{2}) \), find at each point: the curvature and torsion, the vectors of the Frenet frame, and the equations of the osculating plane and the principal circle. Make a sketch.

a) Let \( k \) and \( \tau \) denote respectively the curvature and torsion of a curve in space. Show that, if the curve lies on a sphere, \( \frac{d}{ds} \left( \frac{1}{k \tau^2} \frac{dX}{ds} \right) - \frac{\tau}{k} = 0 \), \( \frac{d}{ds} \) denoting differentiation with respect to arc length along the curve.

b) Give an example of such a curve which does not lie in a plane.

c) Is there a converse to a)? (Hint: For this, look at the center of the osculating sphere to the curve at a point. This sphere is determined by four points on the curve in the same manner that the osculating circle is determined by three.)
Equations of Frenet & Serret

Let $T$, $N$, $B$ be the usual vectors associated with a regular curve $x$ parameterized by arclength $s$.

Already proved are the first & third equations of

\[
\begin{align*}
\frac{dT}{ds} &= N \\ \frac{dN}{ds} &= -TX + B\tau \\ \frac{dB}{ds} &= -N\tau
\end{align*}
\]

To check the second equation:

\[
\frac{dN}{ds} = \frac{d}{ds}(B \times T) = \frac{dB}{ds} \times T + B \times \frac{dT}{ds}
\]

\[
= (-N\tau) \times T + B \times (N\tau) = B\tau - TX.
\]

These, together with $\frac{dT}{ds} = T$, contain all information for the curve.

**Moving frame** or Frenet frame.

or **moving trihedron**.

\[
\Omega(t) = \begin{bmatrix} 0 & -\kappa \tau & 0 \\ \kappa \tau & 0 & -\kappa \tau \\ 0 & \kappa \tau & 0 \end{bmatrix}
\]

Write Frenet equations using matrices:

\[
\frac{d}{dt} [T\,N\,B] = [T\,N\,B] \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}
\]

Note $T$, $N$, $B$, $\kappa$, $\tau$ depend on $s$. 

**Cartan Matrix**
Write the vectors of the frame at
\[ \mathbf{E}(t) = \begin{bmatrix} T & N(t) & B(t) \end{bmatrix} \]
3 x 3 matrix. Vector

Vector fields in moving frame
\( x(t) \) reference curve
\( \dot{x}(t) \) vector field along \( x(t) \)
- vector in forward \( \mathbf{E}(t) \)
\[ \dot{x}(t) = \begin{bmatrix} x'(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \] components of \( \dot{x} \) relative to \( T, N, B \).

Namely,
\[ \frac{dX}{dt} = \lim_{h \to 0} \frac{X(t+h) - X(t)}{h} \]

We have
\[ \frac{dX}{dt} = \frac{d}{dt}(E \dot{X}) = \frac{dE}{dt} \dot{X} + E \frac{d\dot{X}}{dt} \]

\[ \frac{dX}{dt} = E \left( \frac{d\dot{X}}{dt} + \omega \times \dot{X} \right) \]

Motion relative to frame, instantaneous notation about axis in frame

Aside: One can rewrite
\[ \begin{bmatrix} 0 & -\mathbf{\omega} \times & 0 \\ \mathbf{\omega} \times & 0 & -\mathbf{\omega} \\ 0 & \mathbf{\omega} \times & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \mathbf{\omega} \times x_1 \\ 0 \times x_2 \\ \mathbf{\omega} \times x_3 \end{bmatrix} \]

Can be rewritten
\[ \mathbf{\omega} \times \mathbf{\omega} = 0 \]

Canonical coordinates
\[ x(t) \] smooth curve
\[ t \] arc length

Describe curve near \( t = 0 \): use

Maclaurin expansion,
\[ x(t) = x(0) + x'(0) t + \frac{1}{2} x''(0) t^2 + \frac{1}{6} x'''(0) t^3 + \frac{1}{24} x''''(0) t^4 + O(t^5) \]
\[ a(t) = a(0) + t \dot{a}(0) + \frac{1}{2} \ddot{a}(0) t^2 + \frac{1}{6} \dddot{a}(0) t^3 + \frac{1}{24} \ddddot{a}(0) t^4 + O(t^5) \]

We have
\[
T = E \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad T' = E \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},
\]
\[
T'' = E \begin{bmatrix} \Omega^2 + \Omega' \\ 0 \\ 0 \end{bmatrix}, \quad T''' = E \begin{bmatrix} \Omega^2 + 2 \Omega' \dot{\Omega} + \Omega'' \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

Write
\[
x(t) = x(0) + t \dot{x}(0) + \frac{\ddot{x}(0)}{2} t^2 + \frac{\dddot{x}(0)}{6} t^3 + O(t^4)
\]

One obtains (evaluated at \( t = 0 \)):
\[
x(t) = x + \frac{\ddot{x}}{6} t^2 - \frac{\dddot{x}}{8} t^3 + O(t^4)
\]
\[
v(t) = \frac{\dddot{x}}{2} t^2 + \frac{\ddot{x}}{6} t^3 + \frac{\dddot{x}}{24} t^4 + O(t^5)
\]
\[
w(t) = \frac{\dddot{x} \ddot{x} - \dddot{x}}{24} t^4 + O(t^5)
\]

E.g.,
\[
\begin{bmatrix} \Omega^2 + \Omega' \\ \Omega' \end{bmatrix} = \begin{bmatrix} \dot{x}^2 & \dot{x} \ddot{x} \\ \dot{x} \dddot{x} & \dddot{x} \end{bmatrix}
\]

Note \( T''' \) should be checked.

For small \( t \):
\[
v \approx \frac{x}{2} \ddot{x} t^2 \\
w \approx \frac{x \ddot{x}}{6} t^3 \\
w^2 \approx \frac{2x \dddot{x}}{9x} t^3
\]

\[
\begin{array}{c}
\text{OSCILATING} \\
\text{PLANE}
\end{array}
\quad
\begin{array}{c}
\text{RECTIFYING} \\
\text{PLANE}
\end{array}
\quad
\begin{array}{c}
\text{NORMAL} \\
\text{PLANE}
\end{array}
\]

Note

In the small, a curve looks like a twisted secant.
Intrinsic equations of a space curve

The functions $X(t)$ and $T(t)$ are invariants of a space curve - unchanged by rigid motions in $\mathbb{R}^3$.

Conversely, pick curvatures $k(s)$, $\tau(s)$ and a frame $\mathbf{X}(0)$, $T(0)$, $N(0)$, $B(0)$, yields a unique curve - locally.

For:

\[
\begin{align*}
\frac{dX}{ds} &= T \\
\frac{dT}{ds} &= N X \\
\frac{dN}{ds} &= -T X + \alpha B \\
\frac{dB}{ds} &= -N T
\end{align*}
\]

12 first order scalar ODEs - with 12 initial conditions $X(0), T(0), N(0), B(0)$.

Solutions to such a system exist and are unique - locally. See Prob 1.8 - Next Page.

Aside: Solve the Frenet equations, then $\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t T(u) du$.

Uniqueness - another proof.

Assume $X(t)$ and $T(t)$ the same for two curves $\mathbf{X}(t)$ and $\mathbf{X}(t)$ as measured from $\mathbf{X}(0)$ and $\mathbf{X}(0)$, respectively. None curves rigidly so that at $\mathbf{X}(0)$ the frames $\mathbf{X}(0)$, $T(0)$, $N(0)$, $B(0)$ and $\mathbf{X}(0)$, $T(0)$, $N(0)$, $B(0)$ coincide.
Consider the function 

\[ f(t) = T(t), \quad T(t) = N(t) \cdot \vec{v} + B(t) \cdot \vec{z}. \]

Then 

\[ (T \cdot \vec{v})' = N \cdot \vec{v} + T \cdot \vec{h} \]

\[ (N \cdot \vec{v})' = (N + T \cdot \vec{v}) \cdot \vec{v} + N \cdot (-T \cdot \vec{v} + \vec{B}) \]

\[ (B \cdot \vec{v})' = (-N \cdot \vec{v}) \cdot \vec{v} + B \cdot (-T \cdot \vec{v}) \]

All to obtain \( f'(t) = 0 \) all \( t \),

$2B + (t) = constant \quad f(0) = 3 \quad f(t) = 3 \quad all \ t \$. Since \( f(t) \) is the sum of three terms, each term and must be constant by \( 1 \). Hence \( T(t) = T(0) \)

for all \( t \). \( \square \)

**Problem 1.9** Find the "coordinate functions" \( x(t) \) for the cylindrical helix 

 satisfy intrinsic equations are \( x(t) = t/\pi \) and \( t(t) = 1/\pi \). Make a sketch.

**Problem 1.8** Let \( T(t), N(t), B(t) \) be a solution for Frenet's equations.

If \( T, N, B \) are continuous at \( t=0 \), they are continuous for all \( t \).

Hint: \( T, N, B \) are orthogonal exactly when \( E(t) = [T(t), N(t), B(t)] \) is an orthogonal matrix. \( \sigma E = identity \). 

Compute \( E^{-1} \) using \( E' = E \).