10. Abstract surfaces. Theorem of

Preliminary:
- A proof in the case of a simply-connected region.

A formula of Liouville

\[ u, v^2 \] orthogonal
coordinates
\( F = 0 \) as before
\[ \alpha: \begin{cases} u = u'^1 u^1 \\ v = u'^2 u^2 \end{cases} \] curve
\[ l = \text{arc length along } \alpha \]

Field of orthonormal frames:

\[ e_1(u^1, u^2) = \frac{1}{\sqrt{E(u^1, u^2)}} \frac{\partial}{\partial u^1} (u^1, u^2) \]

\[ e_2(u^1, u^2) = \frac{1}{\sqrt{G(u^1, u^2)}} \frac{\partial}{\partial u^2} (u^1, u^2) \]

Unit tangent to \( \alpha \) at \( \alpha \):

\[ T(u) = e_1 \cos \omega + e_2 \sin \omega \]

Unit vector normal to curve and tangent to surface:

\[ U(u) = -e_1 \sin \omega + e_2 \cos \omega \]

Note:

\[ \frac{dT}{ds} = \frac{d}{dt} (e_1 \cos \omega + e_2 \sin \omega) \]

\[ + \frac{d}{dt} (e_1 \sin \omega + e_2 \cos \omega) \]

(Equation continued)
\[
\frac{dT}{dt} = (e_2 \cdot \omega) \cos \tau + e_1 (-\sin \tau) \frac{d\tau}{dt} \\
+ (-e_1 \cdot \omega) \sin \tau + e_2 (\cos \tau) \frac{d\tau}{dt} \\
= (-e_1 \cdot \sin \tau + e_2 \cdot \cos \tau) (\omega + \frac{d\tau}{dt}).
\]

But \( \frac{dT}{dt} = UX \). \( X \) = geodesic curvature.

So

\[
X = \omega + \frac{d\tau}{dt}
\]

\[
(\text{notation of curve}) = (\text{notation of frame}) + (\text{notation relative to frame})
\]

And, \( \mathbf{r} \) replace \( T \) by a (unit) vector \( \mathbf{x} \)

which is parallel along \( \mathbf{x} \). Let

\[
\theta = \text{angle } e_1 \text{ to } \mathbf{x}, \quad \sin \omega + \frac{d\theta}{dt} = 0
\]

(9.9-8), one have

\[
\mathbf{m} = (\text{change in angle around } \mathbf{x}) = \int \frac{d\theta}{dt} d\tau = \int \frac{d\theta}{dt} d\tau = \int_0^l (-\omega) d\tau
\]

\[
l = \text{length of } \mathbf{x}.
\]

**Turning Tangents**

- \( R \): simply connected region within one chain
- \( \partial \): boundary of \( R \)
- \( R \times \partial \): counterclockwise
- \( E \): exterior angles through each
- Tangent to \( \mathbf{x} \) Term at "Vertex"
- \( \theta \): angle \( e_1 (x'(t), x^2(t)) \) to tangent \( T(t) \)
Now, \[ \int_{\partial \Gamma} \frac{d\theta}{ds} ds + \frac{d\theta}{dt} = 2\pi. \]

Valid using any fixed parameter curve on \( \Gamma \).

Essentially the term through \( 2\pi \) once around \( \Gamma \).

And by linearity, \[ \frac{d\theta}{dt} = -\omega + x_2. \]

so
\[ \int_{\partial \Gamma} (-\omega + x_2) ds + \frac{d\theta}{dt} = 2\pi. \]

but
\[ \int_{\partial \Gamma} (-\omega) ds = h = \int_{\partial \Gamma} K ds \text{ by Gauss,} \]

so
\[ \int_{\partial \Gamma} K ds + \int_{\partial \Gamma} x_2 ds + \frac{d\theta}{dt} = 2\pi. \]

This is Gauss–Bonnet for a simply connected region.

Note. This holds for regions needing several patches - by additivity and cancellation, to be shown in general next.

The Theorem of Gauss & Bonnet

Let \( R \) be made up of "triangles", each lying entirely in a single chart - so the formula above holds. The triangular regions \( R_j \) are to be "counterclockwise oriented."

\( S_j \) = boundary of \( R_j \), \( j = 1, 2, 3, \ldots, F \)

\( \theta^1, \theta^2, \theta^3 \) = exterior angles of \( R_j \).
\[ F = \text{number of faces} \]
\[ E = \text{number of edges} \]
\[ V = V_{\text{int}} + V_{\text{bdy}} = \text{number of vertices} \]

For each \( R_j \):

\[
\int_{R_j} k \, dA + \int_{\partial R_j} x_j \, dt + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = 2\pi \sum_{j=1}^{F} \varepsilon_j
\]

Add over \( j = 1, 2, \ldots, E \) noting \( \sum_{j=1}^{F} \int_{R_j} x_j \, dt \) cancel on interior edges:

\[
\int_{R} k \, dA + \int_{R} x_j \, dt + \sum_{j=1}^{F} (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = 2\pi F
\]

To evaluate the sum, use the interior angles \( \pi - \varepsilon_1', \pi - \varepsilon_2', \pi - \varepsilon_3' \) of the triangles. Note:

\[
\sum_{j=1}^{F} \left( (\pi - \varepsilon_1') + (\pi - \varepsilon_2') + (\pi - \varepsilon_3') \right)
\]

\[
= \left( \text{sum of all interior angles} \right)
\]

\[
= \left( \sum_{\text{interior vertices}} + \sum_{\text{boundary vertices}} \right) \left( (\pi - \varepsilon_1') + (\pi - \varepsilon_2') + (\pi - \varepsilon_3') \right)
\]

\[
= 2\pi V_{\text{int}} + \left( \pi V_{\text{bdy}} - \sum \varepsilon \right)
\]

Each contributes \( 2\pi \) to sum each contributes \( \pi - (\text{exterior angle at vertex}) \).
Thus \[ \sum_{V} \left( \varepsilon_{V}^{1} + \varepsilon_{V}^{2} + \varepsilon_{V}^{3} \right) \]
\[ = 3 \pi F, \pi - \left( 2 \pi V_{\text{int}} + \pi V_{\text{bey}} - \Sigma \varepsilon \right) \]
\[ = 3 \pi F - 2 \pi V_{\text{int}} - \pi V_{\text{bey}} + \Sigma \varepsilon, \]
so

\[ \int_{R} k dA + \int_{\partial R} \gamma d\ell + \Sigma \varepsilon \]
\[ = 2 \pi F - \left( 3 \pi F - 2 \pi V_{\text{int}} - \pi V_{\text{bey}} \right) \]
\[ = -2 \pi F + 2 \pi V_{\text{int}} + \pi V_{\text{bey}} \]

**Note:**

\[ \text{(number of boundary edges)} = \text{(number of boundary vertices)}, \]
so \[ 2E = 3F + V_{\text{bey}} \] - counts each edge twice.

Add \[ 2 \pi \left( 3F - 2E + V_{\text{bey}} \right) = 0 \]
to equation above, so

\[ \left( \text{right hand side} \right) = -2 \pi F + 2 \pi V_{\text{int}} + \pi V_{\text{bey}} \]
\[ + 3 \pi F - 2 \pi E + \pi V_{\text{bey}} \]
\[ = 2 \pi (-F + E + V_{\text{int}} + V_{\text{bey}}) \]

But \( X(R) = V - E + F \) is the Euler characteristic of \( R \). So

\[ \int_{R} k dA + \int_{\partial R} \gamma d\ell + \Sigma \varepsilon = 2 \pi X(R) \]

- The general **Cagnon-Boyaart Theorem** for abstract surfaces.
Remarks
1) This shows $X(R)$ does not depend on the triangulation of $R$.
2) Everything is intrinsic -- nothing depends on charts or coordinates employed.
3) The region $R$ can be quite general.
4) If $R$ is orientable, then $X = x + iy$,
   $0 = \gamma(0) = \gamma(1)$,
5) If $R$ is orientable and closed (no boundary), then
   \[
   \int_{R} \mathbf{C} \cdot d\mathbf{A} = 2\pi (2r - 2g).
   \]
6) Total curvature of tangent around a curve $\gamma$,
   \[
   \int_{\gamma} d\mathbf{C} + \mathbf{I} \cdot \mathbf{n} = 2\pi \nu,
   \]
   $\nu$ integer.

To show $\nu = 1$, "flatten" the surface by "homotopy",

\[
E(u, v; \tau) = (1 - \tau) + \tau E(u, v),
\]
\[
G(u, v; \tau) = (1 - \tau) + \tau G(u, v),
\]
$t = 1$ is original surface, $\tau = 0$ is Euclidean plane $d\mathbf{C}^2 = d\mathbf{u}^2 + d\mathbf{v}^2$.
$\nu$ is an integer-valued continuous function of $\tau$, $\nu = 1$ at $\tau = 0$, so $\nu = 1$ for $\tau \geq 1$ also. The total angle $= 2\pi$. 
Applications

1) Geodesic triangles

\[ \int_{R} K dA + (\pi - \alpha) + (\pi - \beta) + (\pi - \gamma) = 2\pi \]

\[ \alpha = \int_{R} K dA = \alpha + \beta + \gamma - \pi. \]

a) If \( K = \text{constant}. \):

\[ \alpha + \beta + \gamma - \pi = K. \text{ (area of } R) \]

Eq: Sphere of radius 1:

\[ \text{area of a spherical triangle} = \alpha + \beta + \gamma - \pi. \]

b) In general: \( \alpha + \beta + \gamma \geq \pi, = \pi, \text{ or } \leq \pi \)

\[ \text{if } K \geq 0, K = 0, \text{ or } K \leq 0. \]

b) Let \( A = \int_{R} dA \) be the area of \( R \).

As \( R \) shrinks to a point \( P \),

\[ \lim_{A \to 0} \frac{1}{A} \int_{R} K dA = K_0 = \text{Gauss curvature at the point } P. \]
Thus:

\[ K_0 = \lim_{A \to 0} \frac{x + y - z}{A} \] (Curve)

Curves converge at \( P \) as limit as vertices of the triangle tend to \( P \). This is Gauss's original definition of curvature.

Note: Shows \( K \) increases.

2) Intersection of geodesics - may about curvature.

Assume \( K < 0 \).

a) For \( \theta \) geodesic
topologically equivalent to \( \pi < \theta \) arc.

For \( \theta \) circular be a geodesic triangle
such that \( \theta + \pi < \pi \). (\( 0 < \theta < 2\pi \)).

b) Two geodesics
does not occur.

This situation cannot occur.

Example:

\[ K = -1 \]

geodesic intersects itself
but does not bound a disc.

Note: Because on the biaxial sphere must be determined later.
3) **Admissible metrics on a surface**

a) Suppose $M$ is a closed surface with $K \equiv 0$. Then $M$ is a torus or Klein bottle.

**Proof:**
\[ \int_M K = 2\pi \chi(M) \quad \text{gives} \quad \chi(M) = 0. \]

b) Suppose $M$ is a closed surface with $K > 0$ everywhere. Then $M$ is a sphere or projective plane.

**Proof:** $\chi(M) > 0$.

**Note:** $K$ must be constant if $M$ is not any topologically a sphere or projective plane.

**Note:** If $M$ is a surface in space, then $M$ is topologically a sphere and is convex relative sense it becomes a convex set in space. (Hadamard).

c) Suppose $M$ is a closed surface with $K < 0$. Then $M$ has genus $g \geq 2$ if it is orientable;

or: $M$ is a projective plane or Klein bottle with at least one boundary if it is non-orientable.

**Proof:** $\chi(M) < 0$.

Proof: Orientable topologically closed surface with $K < 0$ constant and $g \geq 2$ can be constructed using non-Euclidean (Lobachevsky) geometry.

**Note:** Then above a closed...
Example

\[ \begin{align*}
\text{For triangle } \triangle PQR \text{ and line } \alpha: & \quad s = \text{arc length} \\
\int K d\alpha + \left( \int_{\gamma} + \int_{\gamma}^{\phi} + \int_{\gamma}^{\phi} \right) x_1 d\theta \\
& \text{Malus's Law: } \angle \alpha = (\Delta \phi + \pi + \pi) = 2\pi \\
& \text{Exterior angle at tangent point} \\
\int K d\alpha &= \frac{\Delta \phi}{\Delta \theta} = \frac{1}{\Delta \theta} \int_{\gamma}^{\phi} x_1 d\theta \\
& \text{where } \Delta \theta = \int_{\gamma}^{\phi} d\theta. \text{ From figure above, } \\
\int K d\alpha &\approx K \text{ Area (}\Delta \theta \text{)} = O(\Delta \theta^2).
\end{align*} \]

Let \( \theta \to \Phi \) to obtain

\[ \frac{d\phi}{d\theta} = x_1 \]

This extends to abstract surfaces, the elementary definition of the curvature of a curve on a Euclidean plane, and gives a simple description of geodesic curvature in terms of quantities on the surface.