**Lecture 13.** Abstract surface \( M \) \( \text{Vector fields on } M \).

**Vector fields** on \( M \) (with metric).

\[ X \text{ vector field on } M \quad (\text{differentiable}) \]

\[ X = X^1(u,v) \frac{\partial}{\partial u} + X^2(u,v) \frac{\partial}{\partial v} \quad \text{in local coordinates.} \]

**Remark:** An integral curve of \( X \) through \( P = (u_0,v_0) \)

is a solution of the ODE:

\[
\begin{align*}
\frac{du}{dt} &= X^1(u,v) \\
\frac{dv}{dt} &= X^2(u,v) \\
u(t_0) &= u_0, \quad v(t_0) = v_0
\end{align*}
\]

**Singularity**

\( P \) is a singular point of \( X \) if \( X = 0 \) at \( P \).

\[ \text{I.e.} \quad X^1(u_0,v_0) = 0, \quad X^2(u_0,v_0) = 0 \quad \text{for } P = (u_0,v_0). \]

For this case:\n\[ C(t) = P_0 \quad (\text{constant curve}) \text{ is an integral curve of } X. \]

\( P \) is a isolated singularity if points near \( P \) and \( \neq P \) are not singularities of \( X \).
Index

X vector field on M
P isolated singularity of X

\[(u, v)\] coordinate near \(P = (0, 0)\)

\[X_1 = \frac{\partial}{\partial u}, \quad X_2 = \frac{\partial}{\partial v}\]

C small circle in \((u, v)\) coordinate,

\[\text{going once around } P \text{ in } \phi \in \mathbb{R}^+\]

\[\phi \text{ angle } X_1 \text{ to } X \text{ on } C.
\]

\[\omega(\phi) - \omega(\phi_0) = \text{multiple of } 2\pi \quad \text{since } X \text{ and } X_1\]

\[\text{are in same position for } \phi = a \text{ and } \phi = b.\]

\[I_P (X) = \frac{1}{2\pi} (\omega(b) - \omega(a)).\]

Note: 1) \(I_P (X)\) does not depend on choice of coordinates. If \((u', v')\) new, angle between

\[X_1 = \frac{\partial}{\partial u}, \quad \text{and } X'_1 = \frac{\partial}{\partial u'}, \text{ remains unaltered.} \]

\[\omega(b) \text{ and } \omega(a) \text{ which then cancel.} \]
2) \( I_p(X) \) does not depend on choice of \( C \).

Suppose \( C_0 \) and \( C_1 \) are used. In \( u'=u, u''=v \)
consider \( C_\alpha \) \( u' = u'_\alpha(x) \) \((\alpha = 0, 1)\).

Consider \( C_\alpha : u' = (1-\alpha)u'_0(x) + \alpha u'_1(x) \)
for \( 0 \leq \alpha \leq 1 \). \( C_\alpha \) is \( C_0 \) for \( \alpha = 0 \), \( C_1 \) for \( \alpha = 1 \).

Obtain \( \theta(x, \alpha) \) as before. Then

\[ \frac{1}{2\pi i} (\theta(b, \alpha) - \theta(a, \alpha)) \]

is continuous and integrable; hence the

same integer for \( \alpha = 0 \) and \( \alpha = 1 \).

Examples:
Integral curves of \( I_\alpha \) are sketched.

\[ I_p = 2 \quad I_p = 1 \quad I_p = 1 \]

\[ I_p = 1 \quad I_p = 1 \quad I_p = 0 \]
\[ I_p = -1 \quad \text{essentially any function with} \ I_p = -1 \]

\[ I_p = -2. \]

Example: In the Euclidean plane, the field \( X \) with
\[ X^1(x,y) = Re \left( (x + F(y))^n \right) \]
\[ X^2(x,y) = Im \left( (x + F(y))^n \right) \]
has node \( N \) at origin for \( u > 0 \). The field \( X \) with
\[ X^1(x,y) = Re \left( (x - F(y))^n \right) \]
\[ X^2(x,y) = Im \left( (x - F(y))^n \right) \]
has node \( N \) at the origin for \( u > 0 \).

**Total index**

- \( \overline{S} \) closed surface
- \( X \) vector field with only isolated singularities
- \( X \) has only finitely many singularities \( \rho \)
  (by compactness of \( X \)).

\[ I(X) = \sum_{\rho} \frac{I(X)}{\rho} = \text{total index of} \ X. \]

summed over the finitely many singularities \( \rho \) of \( X \).