
**Quadratic surface**

Quadratic surface in space:

\[ z = A x^2 + 2Bxy + Cy^2 \]

or

\[ z = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \]

\[ A, B, C \text{ constants} \]

Rotate through \( \Theta \) in xy-plane:

\[ \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \Theta & -\sin \Theta \\ \sin \Theta & \cos \Theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \]

Obtain \( z = A' x'^2 + 2B' x'y' + C'y'^2 \)

where

\[ \begin{bmatrix} A' & B' \\ B' & C' \end{bmatrix} = \begin{bmatrix} \cos \Theta & -\sin \Theta \\ \sin \Theta & \cos \Theta \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} \cos \Theta & -\sin \Theta \\ \sin \Theta & \cos \Theta \end{bmatrix} \]

Especially

\[ A' + C' = A + C \] Trace invariant

\[ A'C' - B'^2 = AC - B^2 \] Det invariant

and

\[ B' = (C-A) \sin \Theta \cos \Theta + B (\cos^2 \Theta - \sin^2 \Theta) \]

Choose \( \Theta \) so \( \tan 2\Theta = \frac{2B}{C-A} \); then \( B' = 0 \).

Surface has form:

\[ z = a_1 x^2 + a_2 y^2, \quad a_1, a_2 \]

- Elliptic paraboloid
- Hyperbolic paraboloid

**Cases**

- **Elliptic paraboloid**
  \[ AC - B^2 = a_1 a_2 > 0 \]
  \[ a_1 > 0, a_2 > 0 \] Positive definite

- **Hyperbolic paraboloid**
  \[ AC - B^2 = a_1 a_2 < 0 \]
  \[ a_1 < 0, a_2 > 0 \] Negative definite
**Parabolic Cylinder**
\[ AC - B^2 = a_1 a_2 = 0 \]
\[ \text{ranch} = 1 \]
\[ a_1 = 0, \ a_2 > 0 \quad a_1 < 0, \ a_2 = 0 \]

**Plane**
\[ \text{ranch} = 0 \]
\[ a_1 = 0, \ a_2 = 0 \]

**Osculating Paraboloid**

The "best fitting" quadratic surface - as measured from the tangent plane at \( S(u,v) \) (fixed \( u \) and \( v \)) is the osculating paraboloid at \( S(u,v) \).

Taylor expansion at \( S(u,v) \):

\[
S(u+h,v+k) = S(u,v) + S_1 h + S_2 k + \frac{1}{2} \left( S_{11} h^2 + 2 S_{12} h k + S_{22} k^2 \right) + O(3)
\]

where

\[
S_1 = \frac{\partial S}{\partial u}(u,v) \quad S_2 = \frac{\partial S}{\partial v}(u,v), \quad \text{etc.}
\]

**Unit Normal**

To tangent plane:

\[
M(u,v) = \frac{S_1 \times S_2}{||S_1 \times S_2||}
\]

**Distance of surface from tangent plane - at \( S(u,v) \)**

\[
\text{DIST} = (S(u+h,v+k) - S(u,v)) \cdot M(u,v)
\]

\[
= \frac{1}{2} \left( S_{11} h^2 + 2 S_{12} h k + S_{22} k^2 \right) + O(3)
\]

**Osculating paraboloid**

referred to tangent plane:

\[
P = \frac{1}{2} \left( S_{11} h^2 + 2 S_{12} h k + S_{22} k^2 \right)
\]
where $S_{uv} \cdot M = \frac{\partial^2 S}{\partial u \partial v} (u,v) \cdot M(u,v)$, etc.

This gives the "shape" of the surface near $S(u,v)$. Depend on $u$ & $v$.

**Second Fundamental Form**

Quadratic form on the tangent space to surface $S$ at $S(u,v)$:

$$II = (u,v) du^2 + 2(\gamma(u,v) dv^2 + \xi(u,v) dv^2$$

where $\gamma$ at $S(u,v)$:

$$\gamma = S_{uv} \cdot M \quad \eta = S_{uu} \cdot M \quad \xi = S_{vv} \cdot M$$

or

$$II = \sum_{i,j=1}^{2} \gamma_{ij} (u,v) du^i \cdot dv^j$$

where

$$\gamma_{ij}(u,v) = S_{ij}(u,v) = \frac{\partial^2 S}{\partial u^i \partial u^j} \cdot M$$

We have

$$L_{ii} = L, \quad L_{ij} = -L_{ji} \quad \eta_{ii} = \eta, \quad \eta_{ij} = -\eta_{ji}$$

**Note** $\frac{\partial S}{\partial u} \cdot M = 0$ since $\frac{\partial S}{\partial u} = \xi$ is a tangent vector. Then

$$\frac{\partial^2 S}{\partial u \partial u} \cdot M + \frac{\partial S}{\partial u} \cdot \frac{\partial M}{\partial u} = 0$$

so

$$L_{ij} = -\frac{\partial S}{\partial u} \cdot \frac{\partial M}{\partial u}$$

or

$$L = -\frac{\partial S}{\partial u} \cdot \frac{\partial M}{\partial u}$$

$$\eta = -\frac{\partial S}{\partial v} \cdot \frac{\partial M}{\partial v}$$

$$\xi = -\frac{\partial S}{\partial v} \cdot \frac{\partial M}{\partial v}$$

at $S(u,v)$.

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**Curvature of curves on a surface**

$V = u^2$

$W = x^3$

$S(x,y)\rightarrow S(x^3, u^2)$

$\alpha$
\[
\dot{X} = S(x(t)) \quad \text{curve on } S
\]

\[
\frac{d}{dt} S(x(t)) = T_x(x) \cdot \frac{dx}{dt}(t) \quad \text{velocity vector}
\]

\[
T_x = \text{unit tangent to } S \circ \alpha
\]

\[
t = \text{arc length along } S \circ \alpha
\]

\[
\frac{d}{dt} S(x(t)) = S_1(x(t)) \dot{x}'(t) + S_2(x(t)) \dot{x}''(t)
\]

Velocity vector in basis \( S_1, S_2 \)

Differentiate again:

\[
N_x \cdot \ddot{x}^2 + T_x \dddot{x}
\]

\[
= S_{11}(\dot{x}')^2 + 2 S_{12} \dot{x}' \dot{x}'' + S_{22}(\dot{x}'')^2
\]

\[
+ S_1 \ddot{x}' + S_2 \dddot{x} \quad \text{at } S(4,4^2)
\]

Dot normal \( N(x, 4,4^2) \) — the unit normal to \( S \) at \( S(4,4^2) \). Observe that \( T_x, S_1, S_2 \) are tangent to \( S \).

\( (N, N) \cdot \ddot{x}^2 = (S_{11} \cdot N(x,4,4^2)) \cdot (\dot{x}')^2 + 2(S_{12} \cdot N(x,4,4^2)) \cdot \dot{x}' \dot{x}''
\]

\[
+ (S_{22} \cdot N(x,4,4^2)) \cdot (\dot{x}'')^2
\]

Let

\[
\Theta = \text{angle between } N_x \text{ and } N
\]

\[
= \text{angle between principal normal to } S \circ \alpha \text{ and normal to } S
\]

- at \( S(x(t)) \). Depends on point \( (x(t), t) \). Then, using \( L = S_{11} \cdot N, N = S_{12} \cdot N, N = S_{22} \cdot N \) and \( \ddot{x}^2 = E(\dot{x}')^2 + 2F \dot{x}' \dot{x}'' + G(\dot{x}'')^2 \)

we have:

\[
X = \frac{L(\dot{x}')^2 + 2F \dot{x}' \dot{x}'' + G(\dot{x}'')^2}{E(\dot{x}')^2 + 2F \dot{x}' \dot{x}'' + G(\dot{x}'')^2}
\]

Note: The right-hand side depends only on the tangent vector

\[
T_x = S_1 \dot{x}' + S_2 \dot{x}''
\]

To the curve \( S \circ \alpha \) at \( S(x(t)) \).
Consequence: Meusnier's Theorem, 1776. All curves having the same tangent and principal normal at a point of a surface have the same curvature at that point provided that the principal normals are not tangent to the surface.

Remark 1: The last condition means that $\kappa = 0$ if not zero.

2) $K = \kappa$ is the "normal curvature" of the curve - to write later in conjunction with geodesics. It is the component of $\mathbf{S}(\mathbf{W})$ normal to the surface.

Problem 4.1: Find expressions for $L(x,y), L_2(x,y), N(x,y)$ for a surface in the form $z = f(x,y)$.

Problem 4.2: A point $S(u,v)$ on the surface $S$ is called elliptic, parabolic, or hyperbolic according as $LN - N^2 > 0$, $= 0$, or $< 0$. Show that when the parameters are changed from $u,v$ to $u',v'$, the quantity $LN - N^2$ is multiplied by $\left| \frac{D^2(u,v)}{D(u',v')^2} \right|$. Hence, the nature of elliptic, parabolic, and hyperbolic points do not depend on the parameterization.

Problem 4.3: Show that the tangent

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos u \\ \sin u \end{bmatrix} (R + r \cos v) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} (1 - \sin v),$$

$0 < r < R$, contains all three types of points.
Problem 4.4 Find a non-trivial example to illustrate Neumann’s Theorem. The space curve \[ x = a(1 + \cos t), \quad y = a \sin t, \quad z = \frac{a}{2} \] lies on the surface \( x^2 + y^2 + z^2 = a^2 \). Here \( x^2 + y^2 = a^2 \) is a cylinder.

Normal sections

Let \( S = S(u, v) \) be a regular surface. Intersect \( S \) with the plane passing through \( S(u, v) \) \((u, v) \text{ fixed}\) and containing the normal \( N(u, v) \).

Claim: The intersection is a regular curve near \( S(u, v) \).

For we may assume that the surface is in parametric form \( x = x(u, v), y = y(u, v), z = z(u, v) \) with the point in question being \( x=0, y=0 \), and the tangent plane the \( xy \)-plane. (By translation and rotation of coordinates.)

Then \( +, \partial f/\partial x, \partial f/\partial y \) vanish at \( x=0, y=0 \). Then \( S \) meets the plane "spanned" by \[
\begin{bmatrix}
\cos \theta \\
\sin \theta \\
0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]
\text{in } \mathbf{N} =
\begin{bmatrix}
\cos \theta \\
\sin \theta \\
\frac{1}{k} \cos \theta, \sin \theta
\end{bmatrix}.

Then \( \mathbf{N}(u) = \mathbf{N}([\cos \theta, \sin \theta, -t]), \text{ so } \|\mathbf{N}(u)\| \geq 1 \).}

**Curvature**

Consider the normal sections determined by intersecting the plane containing \( N(u, v) \) and \( S(u, v) \times \mathbf{N}(u, v) \) with the surface \( S \). (\textit{Handly are not } \mathbf{x}, \mathbf{y} \text{ there.})
The normal to the curve, \( N \), is in this plane and is normal to \( S \), so \( N = \pm M \) at \( S(u,v) \). Thus, the curvature \( \kappa \) of this normal section is
\[
\pm \kappa = \frac{Lx^2 + 2Mxy + Ny^2}{E(x^2 + 2Fxy + Gy^2)} = \frac{L(u,v)}{E(u,v)}\Gamma(u,v) \quad \text{etc.}
\]

Note: This depends only on \( L, M, N, E, F, G \) at \( S(u,v) \) and on the direction of the vector \( S(u,v)x + S(v,y)y \) (a constant of two homogeneous forms of same degree.)

- use "+" sign - use "-" sign

Once the choice of unit normal \( N \) is fixed, the expression \( \frac{Lx^2 + 2Mxy + Ny^2}{E(x^2 + 2Fxy + Gy^2)} \) will give the \( \kappa \) or \(-\kappa\) as \( u \) and \( v \) vary.

Define:
\[
\kappa_1(u,v) = \frac{\kappa x}{(x,v) = (0,0)} \quad \frac{Lx^2 + 2Mxy + Ny^2}{E(x^2 + 2Fxy + Gy^2)}
\]
\[
\kappa_2(u,v) = \max \quad \text{ditto}
\]
\[
\kappa_2(u,v) = \max \quad \text{ditto}
\]

where \( L, \ldots, G \) depend on \( u, v \).

\( \kappa_1, \kappa_2 \) principal curvatures of \( S \) at \( S(u,v) \)

\( K = \kappa_1 \kappa_2 \) Gaussian curvature

\( H = \frac{1}{2}(\kappa_1 + \kappa_2) \) mean curvature

\( K(u,v) \) and \( H(u,v) \) depend on \( u \) and \( v \).

Next: A method to compute these.
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**Digression - the extreme of the ratio of quadratic forms**

Setting: \(E, F, G, L, M, N\) fixed scalars

The form \(Ex^2 + 2Fxy + Gy^2\) positive definite.

Let \(+(x,y) = \frac{Lx^2 + 2Mxy + Ny^2}{Ex^2 + 2Fxy + Gy^2}\), \((x,y) \neq (0,0)\)

To find:

\[X_1 = \min \{+(x,y)\}, \quad X_2 = \max \{+(x,y)\}\]

over \((x,y) \neq (0,0)\). Since \(+\) is homogeneous of degree 0, one may assume:

\(Ex^2 + 2Fxy + Gy^2 = 1\) and treat this as extreme value constraint:

\[
\begin{cases}
Lx^2 + 2Mxy + Ny^2 = \min \text{ or } \max \\
Ex^2 + 2Fxy + Gy^2 = 1
\end{cases}
\]

Use Lagrange multiplier:

\[
\frac{\partial}{\partial x} \left( Lx^2 + 2Mxy + Ny^2 \right) = \lambda \frac{\partial}{\partial x} \left( Ex^2 + 2Fxy + Gy^2 \right)
\]

\[
\frac{\partial}{\partial y} \left( Lx^2 + 2Mxy + Ny^2 \right) = \lambda \frac{\partial}{\partial y} \left( Ex^2 + 2Fxy + Gy^2 \right)
\]

to obtain:

\[
\begin{bmatrix}
(L - \lambda E)x + (M - \lambda F)y & 0 \\
(N - \lambda F)x + (M - \lambda G)y & 0
\end{bmatrix}
\]

**Facts - easy to prove -**

- The roots of the set \(L - \lambda E \quad N - \lambda F\)
  \(N - \lambda F \quad N - \lambda G\)

  are the extrema \(X_1\) and \(X_2\).

- There are real eigenvalues.

If \(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\) and \(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\) are the eigenvectors for \(X_1\) and \(X_2\), \(i = 1, 2\):

\[
\begin{bmatrix}
L - x_j E & N - x_j F \\
N - x_j F & N - x_j G
\end{bmatrix}
\begin{bmatrix} x_j \\ y_j \end{bmatrix} = 0, \quad j = 1, 2
\]

then:

\[
\begin{bmatrix}
x_j ^2 + 2M x_j y_j + Ny_j^2 = x_j^2 \\
Ex_j^2 + 2Fxy_j + Gy_j^2 = 1
\end{bmatrix}, \quad j = 1, 2
\]

- The last equation is a "normalization"
If \( x_1 \neq x_2 \), then the eigenvalues are orthogonal — with respect to both quadratic forms:

\[
\begin{bmatrix}
x_1^2 \\
x_2^2
\end{bmatrix}
\begin{bmatrix}
E & F \\
F & G
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = 0.
\]

Likewise, for \( x_1 = x_2 \), then \( f(x,y) \) is a constant and that constant is:

\[
x_1 = x_2 = \frac{L}{E} = \frac{F}{G} = \frac{K}{E}.
\]

Therefore, the equation expands to:

\[
\det \begin{bmatrix}
L-\lambda E & E-N \\
M-\lambda E & N-K
\end{bmatrix} = 0
\]

\[
\lambda^2 - \frac{EN-2FN+GL}{EG-F^2} \lambda + \frac{LM-M^2}{EG-F^2} = 0.
\]

The coefficients are the sum & product of \( \lambda \) of \( x_1 \) and \( x_2 \):

\[
x_1 + x_2 = \frac{EN-2FN+GL}{EG-F^2}, \quad x_1 x_2 = \frac{LM-M^2}{EG-F^2}.
\]

Consequence: For a surface \( X = S(u,v) \), with coefficients \( E(u,v),\ldots, L(u,v) \) of the first and second fundamental forms, we have:

\[
K = \frac{LM-M^2}{EG-F^2}, \quad \frac{1}{H} = \frac{1}{2} \frac{EN-2FN+GL}{EG-F^2}.
\]

Curvature, mean curvature — at \( S(u,v) \). These depend on \( u \) & \( v \).

Remark 1) Both \( K \) and \( H \) appear to depend on both the second fundamental form, hence \( K \) can be computed from \( E,F,G \) (and deriv's) alone. The Gaussian curvature \( K \) is, in fact, intrinsic.

2) A surface is a minimal surface when its mean curvature \( H \) vanishes identically.
Classification of points on a surface

1) \( K > 0 \) elliptic point
   i) \( k_1 = k_2 \) umbilic
      characterized by \( \frac{1}{E} = \frac{1}{F} = \frac{N}{G} \)

2) \( K = 0 \)
   i) not all of \( L, M, N \) - parabolic point
      are zero
   ii) all three of \( L, M, N \) - planar point
      are zero

3) \( K < 0 \) hyperbolic point

These terms reflect the shape of the osculating paraboloid at a point. The osculating paraboloid yields more - as in Euler & Dufin - to follow.

Principal directions. At a point \( S(u,v) \) on a surface \( S \), the eigenvectors associated with \( X_1(u,v) \) and \( X_2(u,v) \) give the principal directions.

Denote the unit vectors by \( E_1(u,v) \) and \( E_2(u,v) \). These vectors are tangent to \( S \) but need not be \( \frac{\partial S}{\partial u} \) and \( \frac{\partial S}{\partial v} \) of anything.

Fix \( u \) and \( v \) and rewrite every vector \( X \) tangent to \( S \) at \( S(u,v) \) as
\[
X = S_1(u,v)X^1 + S_2(u,v)X^2, \quad S_1 = \frac{\partial S}{\partial u}, \quad S_2 = \frac{\partial S}{\partial v}.
\]

Likewise express \( Y \). Then
\[
X \cdot Y = \begin{bmatrix} X^1 \mid E \mid F \mid U^1 \\ X^2 \mid F \mid G \mid U^2 \end{bmatrix}.
\]

Especially,
\[
X \cdot X = E(X^1)^2 + 2FX^1X^2 + G(X^2)^2
\]
\[
\Pi(X) = L(X^1)^2 + 2M X^1X^2 + N(X^2)^2.
\]
The curvature of normal sections appear as \( X \cdot X / \Pi(X) \).

Remark. Principal directions at an umbilic or planar point are indeterminate.
Euler's Theorem setting as above.
Use $E_1, E_2$ at $S(\xi, \nu)$ as a basis for the tangent plane.

\[ S_1 = \frac{\partial S}{\partial \xi}, \quad S_2 = \frac{\partial S}{\partial \nu} \]

Write $X = E_1 \cos \alpha + E_2 \sin \alpha$ for an arbitrary unit vector tangent to $S$ at $S(\xi, \nu)$. Then

\[ X \cdot X = 1 \quad \text{unit vector} \]

\[ II(X) = II(E_1) \cos^2 \alpha + II(E_2) \sin^2 \alpha \]

- orthogonality with respect to $II$.

Since $II(E_1) = X_1$ and $II(E_2) = X_2$,
the curvature of the normal section obtained by intersecting $S$ with the plane of $M$ and $X$ is

\[ \pm X = X_1 \cos^2 \alpha + X_2 \sin^2 \alpha. \quad \text{Euler's theorem} \]

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Dubin's indicatrix. The intersection of the osculating parabola with a plane parallel to the tangent plane to a surface at a point gives information about the surface near that point.
At the point \( S(u, v) \) \((u \neq 0 \text{ fixed})\) on a surface \( S \), the Dupin indicatrix is given by \( \Pi(X) = 1 \), where \( X \) ranges over all vectors tangent to \( S \) at \( S(u, v) \). We have six terms of:

- Basis \( S_1 = \frac{\partial S}{\partial u} \) and \( S_2 = \frac{\partial S}{\partial v} \) at \( S(u, v) \):
  \[ X = S_1 X^1 + S_2 X^2 \]
  \[ L(X^1)^2 + 2n X^1 X^2 + n(X^2)^2 = 1 \]

- Basis \( E_1 \) and \( E_2 \) of unit vectors in the principal directions at \( S(u, v) \):
  \[ X = E_1 (3^1)^2 + E_2 (3^2)^2 \]
  \[ k_1 (3^1)^2 + k_2 (3^2)^2 = 1 \] by Euler.

Typical cases are:

- **Elliptic point** \( 0 < X_1 < X_2 \)

- **Hyperbolic point** \( X_1 < 0 < X_2 \)

Problem 4.5: Find the equation of the ellipsoid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \) and prove that tangent planes at these points are parallel to planes which intersect the ellipsoid in circles.
LINES OF CURVATURE

A curve on a surface is a line of curvature if it is tangent at each point to a principal direction. They may be obtained from any ODE.

Diagnosis—continued.
Setting: \( E, F, G, L, M, N \) fixed scalars—as before. Let \( \begin{bmatrix} x \\ y \end{bmatrix} \) be an eigenvector giving a principal direction, and \( \lambda \) the corresponding eigenvalue.

\[
\begin{bmatrix}
  L - \lambda E & \pi - \lambda F \\
  \pi - \lambda F & N - \lambda G
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix} = 0
\]

Expand and eliminate \( x \) to obtain:

\[
\frac{Lx + My}{Ex + Fy} = \frac{\pi x + Ny}{Fx + Gy}
\]

Expand, cancel terms, and rewrite as a determinant:

\[
\begin{vmatrix}
  Gy^2 - kxy & x^2 \\
  E & F & G \\
  L & M & N
\end{vmatrix} = 0
\]

Consequence. One has a quadratic ODE for the lines of curvature on a parameterized surface \( X = S(x, y, z) \):

\[
\begin{vmatrix}
  \frac{E}{L} & \frac{F}{M} & \frac{G}{N} \\
  \pi x^2 + kxyz & x^2 + kxyz & 0 \\
  E & F & G
\end{vmatrix} = 0, \quad E \neq E(x, y, z), \text{ etc.}
\]

Problem 6.6. a) Find the equation of a line of curvature for a surface in range form \( z = f(x, y) \).

b) For the surface \( z = xy \), its equation for lines of curvature is integrable by elementary functions. Use this to parameterize the surface by lines of curvature:

\[
X = \sinh(x + y), \quad Y = \cosh(x - y), \quad Z = xy.
\]
Remark. Principal directions are orthogonal, so parameterizing a surface by lines of curvature gives a "system of orthogonal coordinates". One must avoid umbilics and planar points.

Remark. The equation 
\[ \frac{x^2}{a^2-\lambda} + \frac{y^2}{b^2-\lambda} + \frac{z^2}{c^2-\lambda} = 1 \]

with \( a^2 < b^2 < c^2 \) consists of three families: ellipsoids, hyperboloids of one sheet, and hyperboloids of two sheets. Primes of different families intersect in curves which are lines of curvature in each.

This gives a triply orthogonal system of surfaces for coordinates in space. (See Struik, Ch. 2, Para. 2-11 and 2-16.)

Problem 4.7: Show that the parameter curves \( S(u, \text{const}) \) and \( S(\text{const}, u) \) are lines of curvature if, and only if, both \( \Pi \) and \( \Phi \) are identically zero.

Rodrigues' formula

- a characterization of lines of curvature by the change in \( M(u,v) \) being proportional to the change in \( S(u,v) \):

\[ \Delta M + \kappa \Delta S = 0 \]

along a line of curvature having principal curvature \( \kappa(u,v) \).
Proof A curve of curvature \( X = S(x(t)) \) is characterized by
\[
\frac{d}{dt} S(x(t)) = S_1(x(t)) \dot{x}^1 + S_2(x(t)) \dot{x}^2
\]
being an eigen vector:
\[
\begin{bmatrix}
L - x \dot{E} & \dot{x}^2 \\
\dot{x}^1 & L - x \dot{F}
\end{bmatrix}
\begin{bmatrix}
\dot{x}^1 \\
\dot{x}^2
\end{bmatrix} = 0
\]
on
\[
\sum_{j=1}^{2} \left( L_{ij} - x j_{ij} \right) \dot{x}^j = 0, \quad i = 1, 2
\]
Use \( L_{ij} = \frac{dS}{du} \cdot \frac{dM}{du} \) and \( j_{ij} = \frac{dS}{du} \cdot \frac{dS}{du} \)
to obtain
\[
\sum_{j=1}^{2} \left( -\frac{dS}{du} \cdot \frac{dM}{du} - x \frac{dS}{du} \cdot \frac{dS}{du} \right) \dot{x}^j = 0
\]
for \( i = 1, 2 \). Now, \( \frac{dM}{du} \) and \( \frac{dS}{du} \) are in the plane of \( S_1 \) and \( S_2 \); \( S_1 \) and \( S_2 \) are a basis, so
\[
\frac{dM}{du} + x \frac{dS}{du} = 0 \quad \text{on the curve. QED}
\]
A characterization of the sphere

On a sphere of radius \( a \),
\[
x_1 = \frac{1}{a} \quad \text{since every normal section is a circle of radius } a.
\]
Every point is an umbilic.

Conversely, A regular surface, every point of which is an umbilic, is part of a sphere.

Note the surface is "closed" - compact, connected, and without boundary - the surface is a sphere.
Proof 1) Every point \( S(u,v) \) on the surface \( S \) is an umbilic, i.e., \( \lambda_1(u,v) = \lambda_2(u,v) \).

We may assume these are \( >0 \) and write \( \lambda_1(u,v) \) for \( \lambda_1 \) and \( \lambda_2 \) for \( \lambda_2 \).

2) Every curve on \( S \) is a line of curvature. Especially the coordinate curves \( u = \) constant and \( v = \) constant.

\[
\begin{align*}
\frac{\partial m}{\partial u} + x \frac{\partial s}{\partial u} &= 0 \\
\frac{\partial m}{\partial v} + x \frac{\partial s}{\partial v} &= 0
\end{align*}
\]

By Rodrigues.

Note that \( \lambda_1(u,v) \) is differentiable.

Differentiate the equations above:

\[
\begin{align*}
\frac{\partial}{\partial v} \frac{\partial m}{\partial u} &= -\frac{\partial}{\partial v} \frac{\partial s}{\partial u} - x \frac{\partial}{\partial v} \frac{\partial s}{\partial u} \\
\frac{\partial}{\partial v} \frac{\partial m}{\partial v} &= -\frac{\partial}{\partial v} \frac{\partial s}{\partial v} - x \frac{\partial}{\partial v} \frac{\partial s}{\partial v}
\end{align*}
\]

Since \( \frac{\partial}{\partial v} \frac{\partial m}{\partial u} = \frac{\partial}{\partial v} \frac{\partial s}{\partial u} \), one has

\[
\frac{\partial}{\partial v} \frac{\partial s}{\partial u} = \frac{\partial}{\partial v} \frac{\partial s}{\partial u}
\]

Since \( \frac{\partial s}{\partial u} \) and \( \frac{\partial s}{\partial v} \) are independent

one have

\[
\frac{\partial x}{\partial u} = 0 \quad \text{and} \quad \frac{\partial x}{\partial v} = 0 \quad \Rightarrow \quad x(u,v) = \text{constant} = x.
\]

3) From equations of Rodrigues:

\[
\begin{align*}
\frac{\partial m}{\partial u} + x \frac{\partial s}{\partial u} &= \frac{\partial}{\partial u} (m+xS) = 0 \\
\frac{\partial m}{\partial v} + x \frac{\partial s}{\partial v} &= \frac{\partial}{\partial v} (m+xS) = 0
\end{align*}
\]

so \( m + xS = \text{constant vector} = A \)

Thus \( \| S - \frac{1}{x} A \| = \frac{1}{x^2} \),
so \( S(u,v) \) lies on a sphere of radius \( \frac{1}{x} \)
and center \( \frac{1}{x} A \).

Problem 47. Show: A regular surface, every point of which is a planar point, is part of a plane. Hence, use \( x_1 = x_2 = 0 \) to obtain \( \frac{\partial m}{\partial u} = 0 \) and \( \frac{\partial m}{\partial v} = 0 \).
Remark: A right circular cylinder is locally isometric with a plane, but the surface has no umbilics, only parabolic points. Lines of curvature and related notions are extrinsic.

Problem $\mathbf{4}$: Show that a curve $S(x(t))$ on a surface for which $\frac{\partial}{\partial t} M(x(t))$ is parallel to $\frac{\partial}{\partial t} S(x(t))$ is a line of curvature (converse of Rodrigues).

That is, show $\frac{\partial}{\partial t} M(x(t)) = \lambda(t) \frac{\partial}{\partial t} S(x(t))$, where $|\lambda(t)| = k_1(x(t))$ or $k_2(x(t))$, and $\frac{\partial}{\partial t} S(x(t))$ is the corresponding principal direction.