Compatibility conditions

Motivation. Analogous to the intrinsic eqns of a curve:

given: $g_{ij}(u,v)$, $L_{ij}(u,v)$

find: surface $x = x(u,v)$ with g_{ij}, L_{ij} as coefficients of 1st and 2nd fund. forms.

This means:

$$X_{jk} = \sum_{i=1}^{2} \Gamma^i_{jk} X_i + L_{ij} N$$

must be satisfied – a partial differential eqn.

Remark on PDE's

A PDE need have no solution:

$$\frac{\partial u}{\partial x} = f(x,y,u)$$

$$\frac{\partial u}{\partial y} = g(x,y,u)$$

If $u = u(x,y)$ is a soln: $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

or

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial u} g = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u} f$$ compatibility conditions

– a condition not always satisfied.

But if this condition is satisfied, a solution does exist, in the small, if $u(x_1, y_1)$ is specified. Similarly systems of equations.

<u>Surface</u>. Compatibility conditions will be obtained from

$$X_{ijk} = X_{ikj} \quad \text{or} \quad \frac{\partial}{\partial u^k}\left(\frac{\partial^2 X}{\partial u^j \partial u^i}\right) = \frac{\partial}{\partial u^j}\left(\frac{\partial^2 X}{\partial u^k \partial u^i}\right)$$

with

$$X_{ij} = \sum_{r=1}^{2} \Gamma_{ij}^{r} X_r + L_{ij} N.$$

<u>Weingarten equations</u>. To compute X_{ijk} we will need $N_i = \frac{\partial N}{\partial u^i}$.

Know:

$$N_i = \sum_{r=1}^{2} a_i^r X_r$$

as N_1, N_2 are ~~fell~~ orthogonal to N.

Dot with X_s:

$$-L_{is} = -N_i \cdot X_s = \sum_{r=1} a_i^r X_r \cdot X_s = \sum_{r=1}^{2} g_{rs} a_i^r.$$

Solve for a_i^r:

$$a_i^r = -\sum_{r=1}^{2} g^{rs} L_{ir}.$$

Obtain:

$$N_i = -\sum_{r,s=1}^{2} g^{rs} L_{ir} X_s$$

or

$$\frac{\partial N}{\partial u^i} = -\sum_{r,s=1}^{2} g^{rs} L_{ir} \frac{\partial X}{\partial u^s}.$$

<u>Weingarten equations</u>

<u>Compatibility conditions</u>

Compute : $X_{ijh} = X_{ihj}$.

Omit summation signs - repeated indices denote summation over 1 and 2.

$$X_{ij} = \Gamma_{ij}^{r} X_r + L_{ij} N$$

$$X_{ijh} = \frac{\partial \Gamma_{ij}^{r}}{\partial u^h} X_r + \Gamma_{ij}^{r} X_{rh} + \frac{\partial L_{ij}}{\partial u^h} N + L_{ij} N_h$$

$$= \frac{\partial \Gamma_{ij}^{r}}{\partial u^h} X_r + \Gamma_{ij}^{r} (\Gamma_{rh}^{s} X_s + L_{rh} N) \qquad \text{(Gauss equations)}$$

$$+ \frac{\partial L_{ij}}{\partial u^h} N + L_{ij} (-g^{rs} L_{hr} X_s) \qquad \text{(Weingarten equations)}$$

$$= \left(\frac{\partial \Gamma_{ij}^{r}}{\partial u^h} + \Gamma_{ij}^{s} \Gamma_{sh}^{r} - g^{rs} L_{ij} L_{hs} \right) X_r$$

$$+ \left(\frac{\partial L_{ij}}{\partial u^h} + \Gamma_{ij}^{r} L_{rh} \right) N.$$

Interchange j and h ~~[illegible]~~ and subtract ;
$X_{ijh} - X_{ihj} = 0$ becomes:

$$\frac{\partial \Gamma_{ij}^{r}}{\partial u^h} - \frac{\partial \Gamma_{ih}^{r}}{\partial u^j} + \Gamma_{ij}^{s} \Gamma_{sh}^{r} - \Gamma_{ih}^{s} \Gamma_{sj}^{r} = g^{rs} (L_{ij} L_{hs} - L_{ih} L_{js})$$

$$\frac{\partial L_{ij}}{\partial u^h} - \frac{\partial L_{ih}}{\partial u^j} + \Gamma_{ij}^{r} L_{rh} - \Gamma_{ih}^{r} L_{rj} = 0.$$

Resp. these are : <u>Gauss Theorema Egregium</u> and the <u>Codazzi-Mainardi equations</u>.

Fundamental theorem of surface theory.

The equations

$$\frac{\partial^2 X}{\partial u^i \partial u^j} = \sum_{r=1}^{2} \Gamma_{ij}^{r} \frac{\partial X}{\partial u^r} + L_{ij} N$$

$$\frac{\partial N}{\partial u^i} = - \sum_{r,s=1}^{2} g^{rs} L_{ir} \frac{\partial X}{\partial u^s}$$

~~these all these~~ for the unknown functions $\frac{\partial X}{\partial u^1}, \frac{\partial X}{\partial u^2}, N$ have as their compatibility conditions:

$$\frac{\partial \Gamma_{ij}^{r}}{\partial u^k} - \frac{\partial \Gamma_{ik}^{r}}{\partial u^j} + \Gamma_{ij}^{s} \Gamma_{sk}^{r} - \Gamma_{ik}^{s} \Gamma_{sj}^{r} = g^{rs}(L_{ij} L_{ks} - L_{ik} L_{js})$$

$$\frac{\partial L_{ij}}{\partial u^k} - \frac{\partial L_{ik}}{\partial u^j} + \Gamma_{ij}^{r} L_{rk} - \Gamma_{ik}^{r} L_{rj} = 0.$$

Consequence: If g_{ij}, L_{ij} are prescribed satisfying these equations, a surface exists in the small which has coeffs. for the fundamental forms.

~~N.B. If any g_{ij} are ~~satisfied~~ prescribed, not L_{ij} may exist. An abstract surface may not be imbeddable in three-dimensional space in the small.~~

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Moreover: The surface is unique up to rigid motions.

N.B. If ~~coefficients~~ g_{ij} are given, the L_{ij} must be found — This is a more difficult problem. The answer is yes if either the surface is <u>analytic</u> or C^∞ and K does not change sign.

Codazzi - Mainardi equations.

$$\frac{\partial L_{rj}}{\partial u^k} - \frac{\partial L_{rk}}{\partial u^j} + \Gamma^r_{rj} L_{rk} - \Gamma^r_{rk} L_{rj} = 0$$

One obtains only two equations (with substitutions) for various choices of i, j, k.

Namely: $i=1, j=2, k=1$; and $i=2, j=2, k=1$:

$$\frac{\partial M}{\partial u} - \frac{\partial L}{\partial v} + \Gamma^1_{12} L + \Gamma^2_{12} M - \Gamma^1_{11} M - \Gamma^2_{11} N = 0$$

$$\frac{\partial N}{\partial u} - \frac{\partial M}{\partial v} + \Gamma^1_{22} L + \Gamma^2_{22} M - \Gamma^1_{21} M - \Gamma^2_{21} N = 0$$