Compatibility conditions

Motivation. A generalization to the minimum e.g. of a curve:

- guises: \( g_{ij}(u,v), L_{ij}(u,v) \)
- fields: surface \( x^i(x,u,v) \) with \( g_{ij}, L_{ij} \) as coefficients of 1st and 2nd fund. forms.

This means:

\[
X^i_{\;;jk} = \sum_{r=1}^3 \Gamma^i_{jk} X^r_i + L_{ij} N
\]

must be satisfied — a partial differential eqn.

Remark on PDEs

A PDE need not have a solution:

\[
\frac{\partial u}{\partial x} = f(x,y,u) \\
\frac{\partial u}{\partial y} = g(x,y,u)
\]

If \( u = u(x,y) \) a soln: \( \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \)

or

\[
\frac{\partial f}{\partial y} + \frac{\partial f}{\partial u} g = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u} + \text{compatibility conditions}
\]

— a condition not always satisfied.

But if this condition is satisfied, a solution does exist, i.e. the solution of

\( u(x,y) \) is specified. Similarly systems of equations.
Surfaces. Compatibility conditions must be obtained from

\[ X_{ijh} = X_{i,j} \quad \text{or} \quad \frac{\partial}{\partial u^i} \left( \frac{\partial^2 X}{\partial u^j \partial u^i} \right) = \frac{\partial}{\partial u^j} \left( \frac{\partial^2 X}{\partial u^i \partial u^i} \right) \]

which

\[ X_{ij} = \sum_{r=1}^{n} \Gamma^r_{ij} X_r + L_{ij} N. \]

Weingarten equations. To compute \( X_{ijh} \) we

will need \( N_i = \frac{\partial N}{\partial u^i} \).

\[ X_i = \sum_{r=1}^{n} a_i^r X_r \]

at \( N_1, N_2 \) are \( \perp \) to \( N \).

Dot with \( X_s \):

\[ -L_i s = -N_i \cdot X_s = \sum_{r=1}^{n} a_i^r X_r \cdot X_s = \sum_{r=1}^{n} g_{rs} a_i^r. \]

Solve for \( a_i^r \):

\[ a_i^r = -\sum_{r=1}^{n} g_{rs} L_i r. \]

Obtain:

\[ N_i = -\sum_{s=1}^{n} g_{rs} L_i r X_s \]

or:

\[ \frac{\partial N}{\partial u^i} = -\sum_{s=1}^{n} g_{rs} L_i r \frac{\partial X}{\partial u^s}. \]
Compatibility conditions

Consider \( X_{ijh} = X_{ikj} \).

Omit summation signs - repeated indices denote summation over 1 and 2.

\[
X_{ij} = \Gamma_{ij}^r X_r + L_{ij} N
\]

\[
X_{ijh} = \frac{\partial \Gamma_{ij}^r}{\partial u_h} X_r + \Gamma_{ij}^r X_{rh} + \frac{\partial L_{ij}}{\partial u_h} N + L_{ij} N_h
\]

\[
= \frac{\partial \Gamma_{ij}^r}{\partial u_h} X_r + \Gamma_{ij}^r \left( \Gamma_{rh}^s X_s + L_{rh} N \right) \quad \text{(Gauss equations)}
\]

\[
+ \frac{\partial L_{ij}}{\partial u_h} N + L_{ij} \left( -g^{rs} L_{hr} X_s \right) \quad \text{(Weingarten equations)}
\]

\[
= \left( \frac{\partial \Gamma_{ij}^r}{\partial u_h} + \Gamma_{ij}^r \Gamma_{sh}^r - g^{rs} L_{ij} L_{ks} \right) X_r
\]

\[
+ \left( \frac{\partial L_{ij}}{\partial u_h} + \Gamma_{ij}^r L_{rh} \right) N.
\]

Interchange \( i \) and \( h \) indices and multiply;

\( X_{ijk} - X_{ikj} = 0 \) becomes:

\[
\frac{\partial \Gamma_{ij}^r}{\partial u_h} - \frac{\partial \Gamma_{ih}^r}{\partial u_j} + \Gamma_{ij}^s \Gamma_{sh}^r - \Gamma_{ih}^s \Gamma_{sj}^r = g^{rs} \left( L_{ij} L_{ks} - L_{ik} L_{js} \right)
\]

\[
\frac{\partial L_{ij}}{\partial u_h} - \frac{\partial L_{ih}}{\partial u_j} + \Gamma_{ij}^r L_{rh} - \Gamma_{ih}^r L_{rj} = 0.
\]

Reps these are: Gauss, linear equations and

the Codazzi-Mainardi equations.
Fundamental theorem of surface theory.

The equations

$$\frac{\partial^2 X}{\partial u^i \partial u^j} = \sum_{r=1}^{2} \Gamma^r_{ij} \frac{\partial X}{\partial u^r} + L_{ij} N$$

$$\frac{\partial N}{\partial u^i} = -\sum_{r,s=1}^{2} g^{rs} L_{ir} \frac{\partial X}{\partial u^s}$$

Assumptions for the unknown functions $\frac{\partial X}{\partial u^i}$, $\frac{\partial X}{\partial u^j}$, $N$ base on their compatibility conditions:

$$\frac{\partial \Gamma^r_{ij}}{\partial u^k} - \frac{\partial \Gamma^r_{ik}}{\partial u^j} + \Gamma^s_{ik} \Gamma^r_{sj} - \Gamma^s_{ij} \Gamma^r_{sk} = g^{rs} (L_{rf} L_{ks} - L_{rk} L_{fs})$$

$$\frac{\partial L_{ij}}{\partial u^k} - \frac{\partial L_{ik}}{\partial u^j} + \Gamma^r_{ik} L_{jr} - \Gamma^r_{ij} L_{kr} = 0.$$}

Consequence: If $g_{ij}$, $L_{ij}$ are prescribed satisfying these equations, a surface exists in the small local area under the fundamental forms.

Note: If $g_{ij}$ and $L_{ij}$ are not equal, an abstract surface may not be realizable in three-dimensional space or in the small neighborhood of a point.

Hence: The surface is unique up to rigid motions.
If conditions 1, 2, and 3 are given, the big must be found. This is a more difficult problem. The answer is yes if either the surface is analytic at \( C \) or \( U \) does not change type.

\[ \text{[additional notes and calculations]} \]
Corresponding equations:

\[ \frac{\partial L_{ij}}{\partial u^k} - \frac{\partial L_{ik}}{\partial u^j} + \Gamma_{ij}^r L_{rk} - \Gamma_{ik}^r L_{rj} = 0 \]

One obtains two separate functions (with substitutions) for various choices of \(i,j,k\).

Namely: \(i=1, j=2, k=1\) and \(i=2, j=2, k=1\):

\[ \frac{\partial M}{\partial u} - \frac{\partial L}{\partial V} + \Gamma_{12}^1 L + \Gamma_{12}^2 M - \Gamma_{11}^1 M - \Gamma_{11}^2 N = 0 \]

\[ \frac{\partial N}{\partial u} - \frac{\partial M}{\partial V} + \Gamma_{22}^1 L + \Gamma_{22}^2 M - \Gamma_{21}^1 M - \Gamma_{21}^2 N = 0 \]