\[ v = u^2 \]

**Intrinsic Derivative**

**Curve**: The intrinsic derivative is of vector fields along a given curve on \( S \):

\[ S(\alpha) = S(\alpha^1(t^1), \alpha^2(t^2)) \]

**Note**: \( x = S(\alpha(t)) \) is locally injective, so function on the curve can be written as a function of the parameter \( t \).

**Vector field** — along \( S_0 \alpha \) which is tangent to \( S \):

\[ X(t) = S_1(\alpha(t)) X^1(t) + S_2(\alpha(t)) X^2(t) \]

\[ \frac{DS}{dt}(\alpha(t)) \quad \text{function of } t \]

**Derivatives**

\[ \frac{dX}{dt}(t) = \begin{pmatrix} \text{ordinary "component sense"}\, \text{derivative of } X(t) \text{ in Euclidean } \mathbb{R}^3 \end{pmatrix} \]

**Note**: This vector is not necessarily tangent to the surface.

\[ \frac{DX}{dt}(t) = \begin{pmatrix} \text{projection of } \frac{dX}{dt}(t) \text{ to the tangent plane at } S(\alpha(t)) \end{pmatrix} \]

We have

\[ \frac{DX}{dt} = \frac{dX}{dt} - m(m, \frac{dX}{dt}), \quad m = M(\alpha(t)) \]

**Definition**: \( \frac{DX}{dt} \) is the intrinsic or covariant derivative of \( X \) along \( S_0 \alpha \). Later: \( \frac{DX}{dt} \) depends only on the metric \( g; i; f; g \) and does not need \( \frac{dX}{dt} \).
Calculus: \( \frac{d}{dt} \) along S of x:

1) is linear. For a \& b constants,
\[
\frac{d}{dt}(aX(t) + bY(t)) = a\frac{dX}{dt}(t) + b\frac{dY}{dt}(t).
\]
2) Satisfies Leibniz rule. For \( f \) a function on S of x,
\[
\frac{d}{dt}(f(X(t), Y(t))) = \frac{df}{dt}(X(t), Y(t)) + X(t) \frac{df}{dt}(X(t), Y(t)).
\]
For \( f = X(t) \),
\[
\frac{d}{dt}(X(t)) = \frac{dX}{dt}(t) + X(t) \frac{d}{dt}(t).
\]
Now restrict \( m: \mathbb{R} \to \mathbb{R} \) noting \( m \cdot X = 0 \).
3) Prove the linear product:
\[
\frac{d}{dt}(X(t) \cdot Y(t)) = \frac{dX}{dt}(t) \cdot Y(t) + X(t) \cdot \frac{dY}{dt}(t).
\]
Note: All vectors \( X, \hat{t}, \frac{dX}{dt}, \frac{dY}{dt} \) are tangent to S and so meet only \( \hat{t}, \hat{t} \).

Problem 6.3 Prove 3) above.

Geodesic curvature:
\( S(u(t)) \) curve on surface S.

\( t = \) arc length as parameter,
\( T = \frac{d}{dt} S(u(t)) \) unit tangent.
\( \frac{dT}{dt} = \hat{u} \times g + N \hat{w} \) curvature vector.

Tangential component:
\[
\frac{dT}{dt} = \hat{u} \times g \quad \text{or} \quad \hat{g} = \frac{dT}{dt} \cdot \hat{u}.
\]

Frenet equations: Assume from above.
\[
\begin{align*}
U \cdot U &= 1 \quad \Rightarrow \quad 2U \cdot \frac{dU}{dt} = 0 \\
T \cdot U &= 0 \quad \Rightarrow \quad \frac{dT}{dt} \cdot U + T \cdot \frac{dU}{dt} = 0
\end{align*}
\]
\[
\Rightarrow \quad \frac{dU}{dt} = -T \hat{g}. \quad \hat{g} \quad \frac{dT}{dt} = U \hat{g}
\]
Frenet equations:
\[
\begin{align*}
\frac{dU}{dt} &= -T \hat{g} \quad \frac{dT}{dt} = U \hat{g}
\end{align*}
\]
Exercise 11

Chapter 11

Equation 11.1: \( x \) is a geodesic on \( S \) exactly when \( \frac{\partial}{\partial t} = 0 \) or \( t = \text{const} \).

Exercises

1. Extend to summation convention: reflected
2. indices, one up, one down, are understood and zero
3. for tensor product.
4. yields (BA)^T = B^T A^T
5. for matrix product
6. Raising & lowering indices

\[
\begin{bmatrix}
\tilde{g}_{11} & \tilde{g}_{12} \\
\tilde{g}_{21} & \tilde{g}_{22}
\end{bmatrix}
= \begin{bmatrix}
E & F \\
F & G
\end{bmatrix}
\]

\[
\begin{bmatrix}
\tilde{g}_{11} & \tilde{g}_{12} \\
\tilde{g}_{21} & \tilde{g}_{22}
\end{bmatrix}
^{-1}
= \begin{bmatrix}
\frac{E - F^2}{EG - F^2} & -\frac{F}{EG - F^2} \\
-\frac{F}{EG - F^2} & \frac{G - F^2}{EG - F^2}
\end{bmatrix}
\]

\[
\tilde{g}^{ij} \cdot \tilde{g}^{jk} = \delta^i_j = \begin{cases} 
1 & i = j \\
0 & i \neq j
\end{cases}
\]

Kronecker delta

\[
X_i \text{ vector } \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}, \quad x_i = \tilde{g}^{ij} x^j \text{ covector } \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}
\]

Problem 6.2 For vectors \( X \) and \( Y \), check that

\[
X \cdot Y = \tilde{g}^{ij} x^i y^j = X_a Y^a = X_b Y^b = \tilde{g}^{ab} X_a Y^b
\]

for \( E X^1 X^2 + F (x^1 x^2 + x^2 x^1) + G X^2 X^2 \)

Gauss equations

\( S(u^1, u^2) \) regular surface \( u^1 = u, u^2 = v \)

\[
S_i (u^1, u^2) = \frac{\partial S}{\partial u^i} (u^1, u^2) \quad i = 1, 2
\]

At a point \( S(u^1, u^2) \):

\[
S_1 (u^1, u^2), \quad S_2 (u^1, u^2), \quad N(u^1, u^2)
\]

basis of tangent plane

\( \tilde{\nu} \) normal

is a basis of Euclidean vector space \( \mathbb{R}^3 \).
Write
\[ \frac{\partial S}{\partial u^i} = \sum_{a=1}^2 S_a(u^i, u^2) \Gamma_i^a(u^1, u^2) + M(u^i, u^2) N_i^b(u^1, u^2) \]

Christoffel second fundamental symbols from \( N_i^b \).

On
\[ S_i^b = S_a \Gamma_i^a + M N_i^b \quad i = 1, 2 \]

These are the Gauss equations which define \( \Gamma_i^a \) (and \( N_i^b \)).

Remark: Vectors \( X^i \), forms \( g_{ij} \) and \( Lib \) are "tensors." The Christoffel symbols \( \Gamma_i^a \) are not tensors - they "transform" differently under change of coordinates.

Coordinates for the intrinsic derivative
\[ \begin{cases} u^1 = x^1(u) \\ u^2 = x^2(u) \end{cases} \text{ curve } S(x(u)) \text{ on surface } S. \]

Velocity vector
\[ \frac{d}{dt} S(x(u(t))) = S_1(x^1(u(t)), x^2(u(t))) \Gamma_i^a(u^1, u^2) + S_2(-1-) \frac{d}{dt} x^2(u(t)) \]
\[ = S_i^b \dot{x}^b \text{ summed on } i = 1, 2 \]

Vector field along curve:
\[ X(u(t)) = S_1(x^1(u(t)), x^2(u(t))) X^i(u(t)) + S_2(-1-) \frac{d}{dt} x^2(u(t)) \]
\[ = S_a X^a \text{ summed on } a = 1, 2 \]

Take derivative of \( X(u(t)) \) in \( \mathbb{E}^3 \):
\[ \frac{dX}{dt} = \frac{d}{dt} (S_a X^a) = \left( \frac{d}{dt} S_a \right) X^a + S_a \frac{dX^a}{dt} \]
\[ = \left( S_i^b \dot{x}^i \right) X^b + S_a \dot{X}^a \]
\[ = \left( S_a \Gamma_i^a + M N_i^b \right) \dot{x}^i X^b + S_a X^a \]
\[ = S_a \left( \dot{X}^a + \Gamma_i^a \dot{x}^i X^b \right) + M N_i^b \dot{x}^i X^b \]

Tangent to surface normal to surface
The component of \( \frac{dX}{dt} \) tangent to the surface is:

\[
\frac{dX}{dt} = 5a \left( \frac{dx^a}{dt} + \Gamma^a_{bc} \frac{dx^b}{dt} \frac{dx^c}{dt} \right)
\]

summed over \( i, j, k \) over 1,2

or, the \( \alpha \)-th component is:

\[
\left( \frac{dX}{dt} \right)^\alpha = x^\alpha + \Gamma_{ij}^\alpha x^i x^j
\]

Geodesic:

\[
\begin{align*}
& u^1 = t \quad \text{geodesic} \\ S(x(t)) \quad \text{on} \ S \\
& u^2 = x^2(t) \quad s = \text{arc length}
\end{align*}
\]

\( T(t) = \frac{d}{dt} S(x(t)) = S_\alpha (x(t)) x^\alpha (t) \) \quad \text{unit tangent along } S(x)

\[
\frac{dT}{dt} = S_\alpha \left( x^\alpha + \Gamma_{ij}^\alpha x^i x^j \right)
\]

Thus \( \frac{dT}{dt} = 0 \) gives the system:

\[
\sum_{i,j=1}^2 \Gamma^a_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \frac{du^a}{dt} \frac{du^i}{dt} = 0 \quad \text{for } a = 1, 2
\]

A two coupled non-linear second order ordinary differential equations reduce solution to a geodesic \( u^1 = x^1(t) \), \( u^2 = x^2(t) \) parameterized by arc length.

Problem 6.7 Find equations for geodesics \( S(x(t)) \) leaving parameter \( t \) unchanged, not necessarily arc length.

Hint! \( \frac{dX}{dt} = \frac{dX}{d\tau} \frac{d\tau}{dt} \)
Computation of Christoffel symbols
\[ S_{i6} = S_{a} \Gamma_{i6}^{a} + m \nabla_{i} \nabla_{6} \]
Take inner product with vector \( S_{r} = \frac{\partial S}{\partial u_{r}} \):
\[ S_{r} \cdot S_{i6} = \frac{S_{r} \cdot S_{a} \Gamma_{i6}^{a} + S_{r} \cdot m \nabla_{i} \nabla_{6}}{S_{r}} \]
So
\[ g_{ra} \Gamma_{i6}^{a} = S_{r} \cdot S_{i6} \quad \text{or} \quad \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} \Gamma_{i6}^{1} \\ \Gamma_{i6}^{2} \end{bmatrix} = \begin{bmatrix} S_{1} \cdot S_{i6} \\ S_{2} \cdot S_{i6} \end{bmatrix} \]
and
\[ \Gamma_{i6}^{a} = g_{ra} S_{r} \cdot S_{i6} \quad \text{or} \quad \begin{bmatrix} \Gamma_{i6}^{1} \\ \Gamma_{i6}^{2} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}^{-1} \begin{bmatrix} S_{1} \cdot S_{i6} \\ S_{2} \cdot S_{i6} \end{bmatrix} \]

Christoffel symbols - intrinsic
Begin with \( \Gamma_{i6} = S_{i} \cdot S_{r} \) and take partial derivatives:

\( + \)
\[ \frac{\partial \Gamma_{i6}}{\partial u_{6}} = \frac{\partial S_{i}}{\partial u_{6}} (S_{r} \cdot S_{r}) = S_{6i} \cdot S_{r} + S_{i} \cdot S_{6r} \]

\( - \)
\[ \frac{\partial \Gamma_{i6}}{\partial u_{r}} = S_{r} \cdot S_{6} + S_{i} \cdot S_{6r} \]

\( + \)
\[ \frac{\partial \Gamma_{i6}}{\partial u_{i}} = S_{i} \cdot S_{6} + S_{r} \cdot S_{i6} \]

Add an alternate as indicated:
\[ \frac{\partial \Gamma_{i6}}{\partial u_{6}} - \frac{\partial \Gamma_{i6}}{\partial u_{r}} + \frac{\partial \Gamma_{i6}}{\partial u_{i}} = 2 S_{r} \cdot S_{i6} \]

Using \( \Gamma_{i6}^{a} = g_{ra} S_{r} \cdot S_{i6} \), we have
\[ \Gamma_{i6}^{a} = \frac{1}{2} g_{ra} \left( \frac{\partial \Gamma_{i6}}{\partial u_{6}} + \frac{\partial \Gamma_{i6}}{\partial u_{r}} - \frac{\partial \Gamma_{i6}}{\partial u_{i}} \right) \]
for \( r = 1, 6 \).

Note this says the Christoffel symbols are intrinsic - they depend on the first fundamental form only.
Since $\Gamma^a_{bc} = \Gamma^b_{ac}$, there are just 27 different symbols. In terms of $E_i, F_i, G_i, E_i = \frac{\partial E}{\partial x_i}$ etc., and $W^2 = E_i F_i - F_i^2$, they are

\[
\Gamma^1_{11} = \frac{CE_i - 2FF_i + FE_i^2}{2W^2}, \quad \Gamma^1_{12} = \frac{-FE_i + 2EF_i - EE_i}{2W^2}
\]

\[
\Gamma^1_{12} = \Gamma^1_{21} = \frac{CE_i - FG_i}{2W^2}, \quad \Gamma^1_{13} = \Gamma^1_{31} = \frac{-FE_i + EG_i}{2W^2}
\]

\[
\Gamma^1_{22} = \frac{2CF_i - CG_i - FG_i}{2W^2}, \quad \Gamma^2_{22} = \frac{-2FE_i + FG_i + EG_i}{2W^2}
\]

**Consequence** The intrinsic derivative

\[
\left( \frac{\partial X}{\partial t} \right)^o = \frac{\partial X^o}{\partial t} + \Gamma^o_{i\alpha} \frac{\partial X^i}{\partial t} \times^s
\]

is **intrinsic** - it depends only on the first fundamental form.

**Problem 6.8** Check the formulas above for the Christoffel symbols.

**Example** The plane in polar coordinates.

\[
ds^2 = dr^2 + r^2 d\theta^2, \quad E = 1, F = 0, G = r^2.
\]

One has: $u^1 = u = r, u^2 = v = \theta$.

\[
\Gamma^1_{11} = 0, \quad \Gamma^1_{22} = 0
\]

\[
\Gamma^1_{12} = \Gamma^1_{21} = 0, \quad \Gamma^2_{11} = \Gamma^2_{22} = \frac{1}{r}
\]

\[
\Gamma^1_{22} = -r, \quad \Gamma^2_{22} = 0
\]

The equations for a geodesic are

\[
\begin{align*}
\frac{d^2 r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 = 0 \\
\frac{d^2 \theta}{dt^2} + \frac{1}{r} \frac{dr}{dt} \frac{d\theta}{dt} = 0
\end{align*}
\]
First integral or "conserved quantity":
\[ ds^2 = dr^2 + r^2 d\theta^2 \quad \text{or} \quad \left( \frac{dr}{ds} \right)^2 + r^2 \left( \frac{d\theta}{ds} \right)^2 = 1 \]

Another first integral; second equation times \( r^2 \) is:
\[ r^2 \frac{d^2 \theta}{ds^2} + 2r \frac{dr}{ds} \frac{d\theta}{ds} = 0 \quad \text{or} \quad \frac{dr}{ds} \left( r^2 \frac{d\theta}{ds} \right) = 0 \]

so \( r^2 \frac{d\theta}{ds} = C \), constant.

Divide \( \left( \frac{dr}{ds} \right)^2 + r^2 \left( \frac{d\theta}{ds} \right)^2 = 1 \)

by \( \left( \frac{r^2 d\theta}{ds} \right)^2 = C^2 \)

to obtain \( \frac{1}{r^4} \left( \frac{dr}{ds} \right)^2 + \frac{1}{r^2} = \frac{1}{C^2} \), using \( \frac{dr}{ds} = \frac{dr}{d\theta} \frac{d\theta}{ds} \).

From which:
\[ \pm d\theta = \frac{2r}{r \sqrt{r^2 - 1}} \]

Integrate:
\[ \pm \theta + \text{const} = \arccsc \frac{r}{C} \]

The equation
\[ r = C \sec (\pm \theta + \text{const}) \]

is a straight line in polar coordinates.

Problem 6.9: a) Check that the first equation
\[ \frac{d^2 r}{ds^2} - r \left( \frac{d\theta}{ds} \right)^2 = 0 \]

is also satisfied.

b) Obtain solutions using the first equation and
\[ ds^2 = dr^2 + r^2 d\theta^2 \]

Problem 6.10: Show that the geodesics on a cylinder are helices.

Problem 6.11: Find the equations for geodesics on a surface in range
form \( z = f(x, y) \).