Abstract surfaces, Local motions

Motivation

Quantities involving $E^2$ only which can be distinguished from any choice of parameter are intrinsic and one can ignore the surface being in Euclidean space.

Two-dimensional manifolds

A two-manifold $M$ is a Hausdorff topological space with a family of open sets $(U_i)$ called an atlas satisfying:

1) Each $U_i$ is homeomorphic with an open set in $R^2$ via coordinates $u=u_i^1, v=u_i^2$.
   The $U_i$ are coordinate patches on coordinate neighborhoods called charts.

2) If $U_i \cap U_j \neq \emptyset$ in $M$, then
   \[ U_i = U_i(u_i, v_i) \]  the image of the
   \[ U_j = U_j(u_j, v_j) \] overlap in $R^2$,
   with the functions of class $C^3$, invertible, and $\frac{\partial(u_i, v_i)}{\partial(u_j, v_j)} \neq 0$.

3) $M$ is connected.

4) Any open set $U$ with function $u$ satisfying 1) and 2) is already in the atlas.
   This allows one to introduce new coordinate patches by taking parts.
of old and new without changing the manifold. It suffices to work with a subfamily of charts.

5) \( \pi \) is countable: a countable subfamily of \( \{U\} \) covers \( M \). Thus prevents the manifold from becoming too "large".

Example

The 2-sphere.

Charts are stereographic projection.

Problem 7.1 Let \( M \) be the 2-sphere in \( \mathbb{R}^3 \).

Let \( U \) be a neighborhood of the north pole, \( V \) be the south pole, \( W \) be the plane tangent at the south pole, and let \( \sigma : M \setminus \{ \text{north pole} \} \to \mathbb{R}^2 \) be the stereographic projection.

Find a formula for \( U \circ \sigma^{-1} : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \setminus \{(0,0)\} \).

Curves are described in terms of the coordinates in the charts:

- In \( U \):
  \( u = u(x), \quad v = v(y) \)

For a second chart:

- In \( V' \):
  \( u' = u'(x'), \quad v' = v'(y') \)

On the overlap:

- In \( U \circ \sigma^{-1} \):
  \( u(x) = u(u(x'), v(x')) \)

- In \( V \circ \sigma^{-1} \):
  \( v(x) = v(u(x'), v(x')) \)

A curve is an immersion at the differentiability of \( u(x) \) & \( v(x) \) allows. This is unaltered by a change of coordinate grid in sufficiently differentiable.

Remark \( M \) connected means: any two points of \( M \) can be joined by a continuous curve.
Tangent vectors - at a point.

Give components $X^1$ and $X^2$ of $X$ in each chart.

Write

$$X = X^1 \frac{\partial}{\partial u_1}(u_0, v_0) + X^2 \frac{\partial}{\partial v}(u_0, v_0)$$

to restate as

$$X = \frac{\partial S}{\partial u_1}(u_0, v_0) X^1 + \frac{\partial S}{\partial v}(u_0, v_0) X^2$$

for surface in space. For a second chart

$$\frac{\partial}{\partial u_1} = \frac{\partial u_1}{\partial x^1} \frac{\partial x^1}{\partial u} + \frac{\partial u_1}{\partial x^1} \frac{\partial x^1}{\partial v}$$

- by chain rule

$$\frac{\partial}{\partial v} = \frac{\partial v}{\partial x^1} \frac{\partial x^1}{\partial u} + \frac{\partial v}{\partial x^1} \frac{\partial x^1}{\partial v}$$

- evaluated at approximate point.

Or - as change of basis of tangent vector:

$$\left[ \frac{\partial}{\partial u_1}, \frac{\partial}{\partial v} \right] = \left[ \frac{\partial}{\partial u_1}, \frac{\partial}{\partial v} \right] \left[ \begin{array}{c} \frac{\partial x^1}{\partial u_1} \\ \frac{\partial x^1}{\partial v} \end{array} \right] \left[ \begin{array}{c} \frac{\partial x^1}{\partial v} \\ \frac{\partial x^1}{\partial v} \end{array} \right]$$

For $X = X^1 \frac{\partial}{\partial u_1} + X^2 \frac{\partial}{\partial v}$, we have

$$\left[ \begin{array}{c} X^1 \\ X^2 \end{array} \right] = \left[ \begin{array}{cc} \frac{\partial x^1}{\partial u_1} & \frac{\partial x^1}{\partial v} \\ \frac{\partial x^1}{\partial u_1} & \frac{\partial x^1}{\partial v} \end{array} \right]^{-1} \left[ \begin{array}{c} X^1 \\ X^2 \end{array} \right]$$

at $P_0$.

Cf Problems 3.3 and 4.2.

Remark: One also writes

$$\frac{\partial}{\partial u^r} = \frac{\partial u^r}{\partial u^i} \frac{\partial}{\partial x^i}, \quad X^r = \frac{\partial x^r}{\partial u^i} X^i$$

remains on repeated index, so

$$X = X^i \frac{\partial}{\partial u^i} = X^r \frac{\partial}{\partial u^r}.$$
Velocity vector to a curve at a point

If a curve \( \sigma \) on \( \mathbb{R}^2 \) is given by
\[
\begin{align*}
(u(t), v(t)) = u(t)\, \mathbf{i} + v(t)\, \mathbf{j},
\end{align*}
\]

the derivative
\[
\frac{d\sigma}{dt}(t_0) = \frac{du}{dt}(t_0) \frac{\partial}{\partial u} (u(t_0), v(t_0)) + \frac{dv}{dt}(t_0) \frac{\partial}{\partial v} (u(t_0), v(t_0))
\]

for the velocity vector of \( \sigma \) at \( \sigma(t_0) \). This represents changes of coordinates.

**Problem 7.1**

The coordinates \( u \) and \( v \) are actually functions \( f \) on \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \), defined by
\[
\begin{align*}
(u, v) = f(x, y)
\end{align*}
\]

If \( f : \mathbb{R}^2 \to \mathbb{R} \) is a real-valued function on \( \mathbb{R}^2 \),
recalling \( f(u, v) \) really means \( f(\varphi^{-1}(u, v)) \).

1. Let \( X \) be a vector tangent to \( \mathbb{R}^2 \) at \( x_0 \),
    and let \( \sigma \) be any curve such that \( \sigma(0) = x_0 \) and \( \frac{d\sigma}{dt}(0) = X \).
    Using \( f \circ \sigma = (f \circ \varphi^{-1}) \circ (\varphi \circ \sigma) \), convince yourself
    that \( X \) (at \( x_0 \)) is the "directional derivative" of \( f \) along \( \varphi \circ \sigma \) at \( x_0 \).

\[
\frac{d}{dt} f(\sigma(t)) = \frac{df}{du} (u(t_0), v(t_0)) \frac{du}{dt}(t_0) + \frac{df}{dv} (u(t_0), v(t_0)) \frac{dv}{dt}(t_0)
\]

This is written \( Xf \).

b) Show that \( \mathbb{R}^2 \to \mathbb{R} \) is linear and
\[
X(fg) = (Xf)g + f(Xg).
\]

**These ideas**

Vector tangent to an abstract manifold \( M \) can be obtained as operators called derivations on the real functions on \( M \).
**Vector fields**

A vector field on \( M \) is a map \( p \mapsto X_p \) (a vector tangent to \( M \) at \( p \)).

In coordinates,

\[
X = X^1(u, v) \frac{\partial}{\partial u} + X^2(u, v) \frac{\partial}{\partial v}
\]

with components \( X^1 \) and \( X^2 \), sufficiently smooth.

**Differential equations**

A first order ODE is a vector field \( X \) on \( M \). An integral curve through a point \( p_0 \) is a curve \( \gamma(t) \) satisfying

\[
\begin{align*}
\frac{d\gamma}{dt} &= X_{\gamma(t)} \\
\gamma(0) &= p_0
\end{align*}
\]

In a chart, with coordinates \((x, y)\), these become

\[
\begin{align*}
\frac{dx}{dt} &= X^1(x(t), y(t)) \\
\frac{dy}{dt} &= X^2(x(t), y(t))
\end{align*}
\]

**Abstract surfaces**

- **on two-dimensional Riemannian manifolds**

Equip a two-manifold \( M \) with a smooth field of positive definite quadratic first forms. This is called the metric form.

If \( X \) and \( Y \) are tangent vectors of \( M \) at \( p \), then \( X \cdot Y \) is the product in the metric.

In coordinates \((u, v)\):

\[
E = \frac{\partial}{\partial u} \cdot \frac{\partial}{\partial u}, \quad F = \frac{\partial}{\partial u} \cdot \frac{\partial}{\partial v}, \quad G = \frac{\partial}{\partial v} \cdot \frac{\partial}{\partial v} \quad \text{at } p.
\]

For \( X = X^1 \frac{\partial}{\partial u} + X^2 \frac{\partial}{\partial v} \) at \( p \), we have

\[
X \cdot X = E (X^1)^2 + 2F X^1 X^2 + G (X^2)^2.
\]
or \( ds^2 = \sum_{i,j=1}^n g_{ij}(x^1, x^2) dx^i dx^j \).

This respects change of coordinates.

Quantities that respect change of coordinates—"covariant"—are geometric. Essentially, quantities depending only on the metric are geometric.

**Geometric entities**—cf. "intrinsic".

At one point:
- inner product \( \mathbf{x} \cdot \mathbf{y} \) at same point
- length \( ||\mathbf{x}|| = \sqrt{\mathbf{x} \cdot \mathbf{x}} \) is metric positive definite
- angle \( \cos \angle (\mathbf{x}, \mathbf{y}) = \frac{\mathbf{x} \cdot \mathbf{y}}{||\mathbf{x}|| ||\mathbf{y}||} \)
- orthogonality \( \mathbf{x} \cdot \mathbf{y} = 0 \)

At several points integration is:

- lengths of a curve \( \gamma(t) \): \( \int_a^b \left( \frac{d\mathbf{x}}{dt} \right) \cdot \frac{d\mathbf{x}}{dt} \, dt \)

It may be several charts to cover the curve. For each chart the length of the segment is

\[
\int_a^b \sqrt{E(\mathbf{x}(t), \mathbf{x}(t)) \left( \frac{dx}{dt} \right)^2 + \ldots} \, dt
\]

One adds the length of the segment.

- orthogonal curves \( \frac{dx}{dt} \cdot \frac{dy}{dt} = 0 \)
- orthogonal coordinates \( \frac{\partial^2}{\partial u \partial v} = 0 \) in a chart.

Area element \( dA \) in a chart \( \int dA = \sqrt{E} \, du \, dv \).
At several points - differentiation
- Gaussian curvature - Theorema Egregium
- Intrinsic or covariant derivative along a curve \( C(t), \frac{d}{dt} \):

\[
\frac{dX}{dt}(t) = \left( \frac{dX}{dt} \right) \left( t \right) \frac{d}{du} \left( \frac{d}{dt} \right) \left( C(t), \frac{d}{dt} \right) \frac{\partial}{\partial u} \left( C(t), \frac{d}{dt} \right)
\]

\[
\left( \frac{dX}{dt} \right) \left( t \right) = \frac{d}{dt} \left( t \right) + \sum_{i, q} \gamma_{i, q} \left( C(t), \frac{d}{dt} \right) \frac{du}{dt} \left( C(t), \frac{d}{dt} \right)
\]

- Geodesic curvature

\[
\begin{aligned}
\frac{dT}{dt} &= U \times g \\
\frac{dU}{dt} &= -T \times g
\end{aligned}
\]

Geodesics

\( Tg = 0 \) and \( \frac{dT}{dt} \frac{dt}{dt} = 0 \).

Geodesics as minimizing arc length

- Variation of \( L(g) = \int_a^b \left\| \frac{dX}{dt} \left( t \right) \right\| dt \).

Note: Need to use \( \frac{d}{dt} \) for the variation. And

\[
\frac{d}{d\varepsilon} \int_a^b \frac{dX}{dt} \left( t \right) \cdot \frac{d}{dt} \frac{dX}{dt} \left( t \right) dt
\]

One obtains

\[
\frac{d}{d\varepsilon} \left( L \right) = \left[ \frac{d}{d\varepsilon} \left( t \right) \frac{d}{dt} \left( t \right) \right] \left( t \right) \left. \right|_{t=0}
\]

Example: The Clifford torus

\[
S(1, 1) = \begin{bmatrix}
\cos \theta \\
\sin \theta \\
-\sin \theta \\
\cos \theta
\end{bmatrix}
\]

\[
0 \leq 12 \pi \rightarrow E^4
\]
Take metric induced by Euclidean metric of $E^4$:

\[
\frac{dS}{du} = \begin{bmatrix} \sin u & 0 \\ 0 & \cos u \end{bmatrix}, \quad \frac{dS}{dv} = \begin{bmatrix} 0 & 0 \\ 0 & -\sin v \end{bmatrix},
\]

so

\[
E = \frac{dS}{du} \cdot \frac{dS}{du} = 1, \quad F = \frac{dS}{du} \cdot \frac{dS}{dv} = 0, \quad G = \frac{dS}{dv} \cdot \frac{dS}{dv} = 1.
\]

Thus

\[ds^2 = du^2 + dv^2 \quad \text{and} \quad K = 0 \text{ identically.}
\]

The surface lies in the three-sphere \((x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1\) and divides the 3-sphere into two congruent parts.

Remark. The torus cannot be isometrically embedded in three-sphere \(S^3\), since a closed surface in three-sphere must have nonzero intrinsic Gaussian curvature in positive.

Example. The Poincaré upper half-plane:

\[ds^2 = \frac{dx^2 + dy^2}{y^2} \quad \text{on} \quad y > 0. \]

One chart surface. One checks \(K = -1\) specifically.

Note. This is the "model" of the non-Euclidean plane.

**Theorem of Hilbert.** The abstract surface above can now be isometrically embedded in three-dimensional Euclidean space.

**Problem (7.3)** Show that the geodesics in the Poincaré upper half-plane are semi-circles and lines orthogonal (Euclidean) to the \(x\)-axis.
Geodesic polar coordinates

Abstract

Fix a point 0 (origin) and a direction (θ = 0) from which to measure angles at 0. Obtain geodesic polar coordinates (r, θ) near 0 by measuring the distance along the geodesic through 0 having angle θ. In these coordinates

\[ ds^2 = E(r, \theta) dr^2 + 2F(r, \theta) dr d\theta + G(r, \theta) d\theta^2. \]

It is, in fact, cleaner than this.

a) On a geodesic r = const, \( \frac{dr}{dt} = 0 \) \( \Rightarrow E(r, \theta) = 1 \) if constant.

b) Examine the equations for a geodesic:

\[ \frac{d^2 r}{dt^2} + \Gamma^1_{11} \left( \frac{dr}{dt} \right)^2 + 2 \Gamma^1_{12} \frac{dr}{dt} \frac{d\theta}{dt} + \Gamma^1_{22} \left( \frac{d\theta}{dt} \right)^2 = 0 \]

\[ \frac{d^2 \theta}{dt^2} + \Gamma^2_{11} \left( \frac{dr}{dt} \right)^2 + 2 \Gamma^2_{12} \frac{dr}{dt} \frac{d\theta}{dt} + \Gamma^2_{22} \left( \frac{d\theta}{dt} \right)^2 = 0 \]

The radial lines \( r = r \), \( \theta = \text{const} \), are geodesics, so

\[ \Gamma^1_{11} = 0 \quad \text{and} \quad \Gamma^2_{11} = 0. \]

Now

\[ \Gamma^1_{11} = \frac{GE_1 - 2EF_1 + F^2}{2(EG - F^2)}, \quad \Gamma^1_{22} = \frac{-FF}{G + F^2} = 0 \]

\[ \Gamma^2_{11} = \frac{-FE_1 + 2EF_1 - F^2}{2(EG - F^2)}, \quad \Gamma^2_{22} = \frac{F_1}{G - F^2} = 0. \]

Thus, \( F_1 = \frac{dF}{dr} \geq 0 \) is zero. \( F = F(r, \theta) \) depends on \( \theta \) only.

c) Observe,

\[ S_1 = \frac{\partial}{\partial \theta} (r(\theta)) \]

is unique vector tangent to radial geodesic.
5_2 = \frac{D}{D\theta} \left( \eta \right) \text{ it velocity vector.}

As a velocity vector,
\lim_{r \to 0} S_2 = 0, \text{ since }
0 \leq \theta \leq 2\pi \text{ and lengths }
g_1 \to 0 \text{ as } r \to 0.

Thus \( F(\theta) = S_1 \cdot S_2 \to 0 \text{ or } r \to 0 \).

Since \( F \) shall not depend on \( r \),
\( F(\pi \theta) = 0 \text{ for all } \pi \theta. \)

Consequence! The geodesic polar coordinates,
\( dt^2 = dr^2 + g(\pi \theta) d\theta^2. \)

Curvature: We need to analyze \( G \) near \( r = 0. \)

a) On a geodesic circle of radius \( r \),
\( \theta = \sqrt{G(\pi \theta)} \theta, \text{ so } \)
\[ \text{(length of geod. circle)} = \int_0^{2\pi} \sqrt{G(\pi \theta)} d\theta \]
As \( r \to 0 \), this length \( \to 0 \), so
\[ \lim_{r \to 0} \sqrt{G(\pi \theta)} = 0 \]

b) On a geodesic circle!
\[ \text{(length of geod. arc )} = \int_0^\pi \sqrt{G(\pi x)} dx \]
\[ = \sqrt{G(\pi \theta + \eta)} \eta \quad \text{by integral mean value theorem} \]
\[ = ( \sqrt{G(\pi \theta + \eta)} - \sqrt{G(0, \theta + \eta)} ) \eta \]
\[ = \frac{D\sqrt{G}}{D\eta} (\pi, \theta + \eta) \eta \text{ - by differential mean value theorem} \]

Note \( \eta \) is dependent on \( r, \theta. \).
The value of this integral should be approximately 1, i.e.

\[ \lim_{r \to 0} \frac{1}{r^4} \int_0^{2\pi} \sqrt{G(r, \theta)} \, d\theta = 1 \]

or

\[ \lim_{r \to 0} \frac{\sqrt{G}}{r} (r, \theta) = 1. \]

1) Formula for curvature (Prob. 5.5):

\[ K(r, \theta) = \frac{1}{\sqrt{G}} \frac{d^2 \sqrt{G}}{dr^2} \]

Set \( K_0 = K(0, \theta) \) (same for all \( \theta \)).

9) To establish a formula for \( G \):

\[ \sqrt{G(r, \theta)} = \sqrt{G(0, \theta)} + \frac{d \sqrt{G}}{dr} (0, \theta) r + \frac{1}{2} \frac{d^2 \sqrt{G}}{dr^2} (0, \theta) r^2 + \frac{1}{6} \frac{d^3 \sqrt{G}}{dr^3} (0, \theta) r^3 + O(r^4) \]

Note that \( \theta \) enters the estimate in \( O(r^4) \).

Now

\[ \frac{d^2 \sqrt{G}}{dr^2} (r, \theta) = -K(r, \theta) \frac{d \sqrt{G}}{dr} (r, \theta) \]

and

\[ \frac{d^3 \sqrt{G}}{dr^3} = \frac{dK}{dr} \frac{d \sqrt{G}}{dr} - K \frac{d^2 \sqrt{G}}{dr^2} \]

This means \( \sqrt{G}(0, \theta) = 0 \), \( \frac{d \sqrt{G}}{dr}(0, \theta) = 1 \) yields

\[ \sqrt{G(r, \theta)} = r - \frac{K_0}{6} r^3 + O(r^4) \]

and

\[ G(r, \theta) = r^2 - \frac{K_0}{6} r^4 + O(r^5) \].

Geodetic circle

- constant curvature & area

\[ L(r) = \int_0^{2\pi} \sqrt{G(r, \theta)} \, d\theta \]

\[ = \int_0^{2\pi} (r - \frac{K_0}{6} r^3 + O(r^4)) \, d\theta \]

\[ = 2\pi \left( r - \frac{K_0}{6} r^3 \right) + O(r^4) \]
\[ \frac{2 \pi r - L(r)}{\frac{\pi r^3}{6}} = K_0 + O(r) \]

and
\[ K_0 = \lim_{r \to 0} \frac{2 \pi r - L(r)}{\frac{\pi r^3}{6}} \]

And
\[ A(r) = \iint \sqrt{G(\rho, \theta)} \, d\rho d\theta \]
\[ \text{where } 0 \leq \rho \leq r, \quad 0 \leq \theta \leq 2\pi \]
\[ = \iint \left( \rho - \frac{K_0}{6} \rho^3 + O(\rho^5) \right) \, d\rho d\theta \]
\[ = 2\pi \left( \frac{r^2}{2} - \frac{K_0}{12} r^4 + O(r^6) \right) + O(r^5) \]
\[ \to \frac{\pi r^2 - A(r)}{\frac{\pi \gamma^2}{12}} = K_0 + O(r) \]

and
\[ K_0 = \lim_{r \to 0} \frac{\pi r^2 - A(r)}{\frac{\pi \gamma^2}{12}} \]

Remark: These two formulae show how to find curvature intrinsically - geometrically.

Also, if \( K_0 \) is \( >0 \) or \( <0 \), circumference and area either fall short of or exceed their Euclidean values.

These formulae show - by "hence much."

**Surface of constant curvature - local.**

**Theorem:** Any two surfaces having the same constant Gaussian curvature are isometric in the small,
Proof. Case that \( K = -\frac{1}{h^2} \) is negative.

The formula \( K = -\frac{1}{V G} \frac{D^2 V G}{D r^2} \) gives

due initial value problem (for any \( \theta \) fixed):

\[
\begin{cases}
\frac{D^2}{D r^2} V G + \frac{1}{h^2} V G = 0 \\
V G = 0 \\
\frac{D V G}{D r} = 1
\end{cases}
\at r = 0
\]

The solution is \( V G = k \) which \( \frac{r}{h} \).

The positive and zero curvature cases are analogous. \( \Box \).

**Problem 7.4** Verify - as in the theorem - that one obtains the three “models”

due metric:

\[
\begin{align*}
&dr^2 + (a \sin \frac{r}{a})^2 d\theta^2, \quad K = \frac{1}{a^2} \\
&rs^2 + r^2 d\theta^2, \quad K = 0 \\
&dr^2 + (h \sin \frac{r}{h})^2 d\theta^2, \quad K = -\frac{1}{h^2}
\end{align*}
\]
Properties of the intrinsic curvature.

\( C(x), a \leq x \leq b \)

\( V(x), \; W(x) \) vector

curve on an abstract surface \( M \)

1) \( \frac{D}{dx} (V + W) = \frac{DV}{dx} + \frac{DW}{dx} \)

\( \frac{D}{dx} (aV) = a \frac{DV}{dx} \quad a = \text{constant} \)

Proof is trivial.

2) \( \frac{D}{dx} (fV) = \frac{df}{dx} V + f \frac{DV}{dx} \), \( f = f(x) \) differentiable function along \( C \).

\( \left( \frac{D}{dx} (fV) \right)' = \frac{df}{dx} (fV)' + \sum \Gamma_{rs}^{i} (fV') \frac{dV_r}{dx} \)

\( V \cdot W = \frac{df}{dx} V' + f \left( \frac{dV'}{dx} + \sum \Gamma_{rs}^{r} V_r \frac{dW_s}{dx} \right) \)

\( = \left( \frac{df}{dx} V + f \frac{DV}{dx} \right)' \quad i = 1, 2. \)

3) \( \frac{d}{dx} <V, W> = <\frac{dV}{dx}, W> + <V, \frac{dW}{dx}> \)

Proof. \( <V, W> = \sum \tilde{g}_{ij} V^{i} W^{j}. \)

Will need \( \frac{\partial \tilde{g}_{ij}}{\partial u^{k}} \).

\( \Gamma_{jk}^{i} = \frac{1}{2} g^{ir} \left( \frac{\partial \tilde{g}_{jk}}{\partial u^{r}} + \frac{\partial \tilde{g}_{jr}}{\partial u^{k}} - \frac{\partial \tilde{g}_{kr}}{\partial u^{j}} \right) \) (symmetric under \( i \))
\[
\frac{\partial g_{jk}}{\partial u^k} + \frac{\partial g_{jh}}{\partial u^i} - \frac{\partial g_{jh}}{\partial u^i} = 2 \, g_{jr} \Gamma^r_{jh} \\
\frac{\partial g_{jk}}{\partial u^n} + \frac{\partial g_{kn}}{\partial u^i} - \frac{\partial g_{kn}}{\partial u^i} = 2 \, g_{jr} \Gamma^r_{kn} \quad \text{(interchanged)}
\]

Assume indices by \( \delta \):

\[
\frac{\partial g_{jk}}{\partial u^i} = g_{jr} \Gamma^r_{jk} + g_{jr} \Gamma^r_{kj}.
\]

\[
\frac{d}{dt} \langle V, W \rangle = \frac{d}{dt} \left( g_{ij} V^i W^j \right) \quad \text{(summations)}
\]

\[
= g_{ij} \frac{dV^i}{dt} W^j + g_{ij} V^i \frac{dW^j}{dt} + g_{ij} V^i \frac{dW^j}{dt}.
\]

\[
= g_{jr} \Gamma^r_{jk} \frac{dV^i}{dt} W^j + g_{jr} \Gamma^r_{kj} \frac{dV^i}{dt} W^j \\
+ \cdots + \cdots
\]

\[
= g_{jr} \left( \frac{dV^i}{dt} + \Gamma^r_{kj} V^i \frac{dV^j}{dt} \right) W^j + g_{jr} V^i \frac{dW^r}{dt}.
\]

\[
= g_{jr} V^i \left( \frac{dV^r}{dt} + \Gamma^r_{kj} \frac{dW^j}{dt} \right) W^j \\
+ g_{jr} V^i \frac{dW^r}{dt}.
\]

\[
= \langle \frac{dV}{dt}, W \rangle + \langle V, \frac{dW}{dt} \rangle.
\]

Note: For surface in space rotation can be verified using

\[
\frac{dV}{dt} \cdot W = \frac{dV}{dt} \cdot W \quad \text{etc., as } W \text{ is tangent}.
\]