Appendix: Geodesic polar coordinates via the exponential map.

DE of geodesics

\[ \frac{d}{dt} \left( \frac{d \mathbf{C}}{dt} \right) = 0 \quad \text{or} \quad \frac{d^2 \mathbf{u}}{dt^2} + \frac{1}{\mathbf{r}} \sum_{\mathbf{j}, \mathbf{k}=1}^{\mathbf{r}} \frac{d \mathbf{u}_j}{dt} \frac{d \mathbf{u}_k}{dt} = 0 \]

preferential to arc length.

Fix \( P_0 \), first point a case of \( P_0 \) \( u^1 = 0 \), \( u^2 = 0 \)

For any \((x^1, x^2)\) where \( X = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} \)

tangent to \( M \) at \( P_0 \), let \( \Phi(x^1, x^2) \) be the solution of the DE for which \( \Phi(0, x) = P_0 \)

and \( \frac{\partial \Phi}{\partial x} (0, x) = X \). In cases:

\[ \Phi''(0, x^1, x^2) = 0, \quad \frac{\partial \Phi''}{\partial x^1} (0, x^1, x^2) = x^1, \quad r = 1, 2. \]

Consider \( \Phi(x^1; x^2) \). \quad \frac{d}{dt} \left( \frac{d \Phi}{dt} (x^1; x^2) \right) = \alpha^2 \frac{d}{dt} \frac{d}{dx} \Phi(x^1; x^2) = 0; \quad (\Phi(x^1; x^2))_{x=0} = 0 \quad \text{and} \quad \left( \frac{\partial \Phi}{\partial x} (0, x) \right)_{x=0} = \alpha X.

So:

\[ \Phi(x^1; x^2) = \Phi(x^1; \alpha X). \]

Exponential map

\[ e^{\Phi}(X) = \Phi(1; X) \quad (e = e^{\Phi_{P_0}}(X)) \]

\[ \{ \text{tangent vectors} \} \rightarrow \{ M, \text{near } P_0 \} \]

\[ X \rightarrow \text{ (geodesic from } P_0 \text{ with } \frac{dx}{dt} = X) \]

\[ X \rightarrow e^{\Phi_{P_0}} X \]
Note: $f$ is differentiable since it comes from $O\mathbb{E}$.

Jacobian of $f$ at $x^1 = 0$, $x^2 = 0$:

$$Q^f(1; x^1, x^2) = \frac{\partial Q^f}{\partial x^1} (1; x^1, x^2)$$

where:

$$\frac{\partial Q^f}{\partial x^i} (1; x^1, x^2) = \sum_{r=1}^{2} \frac{\partial Q^f}{\partial x_r} (1; x^1, x^2) x_r^2$$

$\mathbf{r} = 0$:

$$x^1 = \frac{\partial Q^f}{\partial x^1} (0; x^1, x^2) = \sum_{r=1}^{2} \frac{\partial Q^f}{\partial x_r} (1; 0, 0) x_r^2$$

So:

$$\frac{\partial Q^f}{\partial x^1} (1; 0, 0) = 0$$

Thus, near $P_0$, $(x^1, x^2)$ can be used as co-ords.

NB: $f$ only makes lines $a x^1 + b x^2 = 0$ for geodesics.

Polar co-ords:

Choose $u^1, u^2$ to be orthonormal parameters: $\mathbf{r} = 0$.

Can assume $\frac{\partial}{\partial u^1}$, $\frac{\partial}{\partial u^2}$ unit vectors at $P_0$: $E = 1$,

$G = 1$ at $P_0$. Further $x^1 = ru^1 \
\qquad x^2 = ru^2$.

Then:

$$\exp \left( r \cos \theta \frac{\partial}{\partial u^1} + r \sin \theta \frac{\partial}{\partial u^2} \right) = P$$

$P$ is a geodesic in $\mathbb{E}$ at $P_0$.

For fixed $\theta$, $P$ has a distance $r$ from $P_0$ geodesically along a geodesic at $r = 0$.

N.B. Simplicity of the coordinates in no way

... 

\begin{align*}
\text{Polar co-ords:} & \quad x^1 = ru^1, \quad x^2 = ru^2 \\
\text{Can plane:} & \quad \frac{\partial f}{\partial x^1} = \frac{\partial f}{\partial x^2} \left( \text{at } P_0 \right) \quad \text{differentiable.}
\end{align*}