Appendix Geodesic polar coordinates via the exponential map.

DE of geodesics

$$\frac{D}{dt}\left(\frac{dC}{dt}\right)=0 \quad \text{or} \quad \frac{d^2u^i}{dt^2}+\sum_{j,k=1}^{2}\Gamma^i_{jk}\frac{du^j}{dt}\frac{du^k}{dt}=0$$

t proportional to arc length.

Fix P_0, ~~[illegible]~~ coords of P_0 $u^1=0,\ u^2=0$

For any (any length) vector $X = x^1\frac{\partial}{\partial u^1}+x^2\frac{\partial}{\partial u^2}$ tangent to M at P_0 let $\Phi(t;X)$ be the solution of the DE for which $\Phi(0;X)=P_0$ and $\frac{d\Phi}{dt}(0;X)=X$. In coords: $u^i=\varphi^i(t;x^1,x^2)$.

$\varphi^i(0;x^1,x^2)=0$, $\frac{d\varphi^i}{dt}(0;x^1,x^2)=x^i$ $\quad i=1,2.$

Consider $\Phi(\alpha t;X)$. $\frac{D}{dt}\left(\frac{d}{dt}\Phi(\alpha t;X)\right)=\alpha^2\frac{D}{dt}\frac{d}{dt}\Phi(t;X)$ $=0$; $\left(\Phi(\alpha t;X)\right)_{t=0}=0$ and $\left(\frac{d}{dt}\Phi(\alpha t;X)\right)_{t=0}=\alpha X$.

So: $\Phi(\alpha t;X)=\Phi(t;\alpha X)$.

Ex Exponential map

$$\exp(X) = \Phi(1;X) \qquad (\text{or } \exp_{P_0}(X))$$

$$\left\{\begin{matrix}\text{tangent vectors}\\ \text{at } P_0 \text{ to } M\end{matrix}\right\} \longrightarrow \{M, \text{ near } P_0\}.$$

$$X \longrightarrow \left(\begin{matrix}\text{geodesic from } P_0 \text{ with initial}\\ \text{velocity } X \text{ followed to } t=1\end{matrix}\right)$$

Note: exp is differentiable since it comes from DEs.

Jacobian of exp at $x^1=0,\ x^2=0$:

$$\varphi^{i'}(t;x^1,x^2) = \varphi^{i'}(1;tx^1,tx^2)$$

$$\frac{d\varphi^{i'}}{dt}(t;x^1,x^2) = \sum_{r=1}^{2} \frac{\partial\varphi^{i'}}{\partial x^r}(1;tx^1,tx^2)\,x^r$$

$t=0$:

$$x^{i'} = \frac{d\varphi^{i'}}{dt}(0;x^1,x^2) = \sum_{r=1}^{2} \frac{\partial\varphi^{i'}}{\partial x^r}(1;0,0)\,x^r.$$

So: $\dfrac{\partial\varphi^{i'}}{\partial x^r}(1;0,0) = \delta^{i'}_r$, $\left(\begin{matrix}\text{Jacobian}\\ \text{determinant}\end{matrix}\right) = 1 \neq 0.$

Thus near P_0, (x^1,x^2) can be used as coords.

N.B. exp maps lines $a_1x^1+a_2x^2=0$ to geodesics.

Polar coords.

Choose u^1,u^2 to be orthogonal parameters: $F\equiv 0$. Can assume $\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}$ unit vectors at P_0: $E=1$, $G=1$ at P_0. Introduce $x^1 = r\cos\theta$, $x^2 = r\sin\theta$

Then: $\exp\left(r\cos\theta\,\frac{\partial}{\partial u^1} + r\sin\theta\,\frac{\partial}{\partial u^2}\right) = P$ and

P is assigned coordinates (r,θ) (at P_0). For fixed θ, P lies a distance r from P_0 ~~geodesic~~ along a geodesic; at $r=0$ the initial tangent to this geodesic makes an angle θ with $\frac{\partial}{\partial u^1}$ at P_0.

N.B. "Singularity" of the coordinates are no more than polar coordinates in the plane.

$$\left\{\begin{matrix}\text{polar coordinates}\\ \text{in plane}\end{matrix}\right\} \xrightarrow{\substack{x^1=r\cos\theta\\ x^2=r\sin\theta}} \left\{\begin{matrix}\text{tangent vectors}\\ \text{at } P_0 \text{ to } M\end{matrix}\right\} \xrightarrow[\text{differentiable}]{\exp} M.$$