$\boxed{A}$. Additional problems

Problem (A.1) Let α be the trefoil knot

$$\alpha(t) = \begin{bmatrix} \cos 2t \\ \sin 2t \\ 0 \end{bmatrix} (5 + 4\cos 3t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} 4 \sin 3t.$$

See Problem 2.2.

a) Find the curvature and torsion at the points where the curvature is maximum or minimum. By symmetry it suffices to do one of each. How many are there?

b) Find the (equations of) the osculating plane and circle at the points in part a).

c) Find (numerically) the total curvature of α.

d) Make an accurate sketch.

Problem (A.2) a) Let $\alpha(t)$ be a curve in space and $\gamma(t) = \alpha(t) + N(t)\frac{1}{\kappa(t)}$ the locus of its centers of curvature. Show that the tangents to α at $\alpha(t)$ and the tangent to γ at $\gamma(t)$ are orthogonal.

b) Illustrate a) with the twisted cubic $\alpha(t) = \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}$. Draw carefully an appropriate figure.

Problem (A.3) Prove that a closed regular surface in three-dimensional space cannot have Gaussian curvature that is everywhere negative. Prove that it must have points of positive Gaussian curvature.

<u>Problem</u> (A.4) a) Show that a curve $\sigma(t) = S(\alpha(t))$ on a surface S is a geodesic if and only if

$$n(\alpha(t)) \times \dot{\sigma}(t) \cdot \ddot{\sigma}(t) = 0,$$

where n is the (unit) normal to S at $\sigma(t)$ and "$\cdot$" denotes d/dt.

b) Use a) to obtain the geodesics on a sphere in Euclidean space.

<u>Problem</u> (A.5) Consider the torus

$$S(u,v) = [(b + a\cos u)\cos v, (b + a\cos u)\sin v, a\sin u],$$

where $0 < a < b$, in Euclidean space.

i) Express the equations for a geodesic other than a parameter curve as

$$\frac{du}{dv} = (\text{function of } u)$$

ii) Prove that a geodesic which cuts the curve $u=0$ at an angle α $(0 < \alpha < \frac{\pi}{2})$ will also meet the curve $u=\pi$ provided that $\cos\alpha < \frac{b-a}{b+a}$.
What happens for $\cos\alpha = \frac{b-a}{b+a}$ and $\cos\alpha > \frac{b-a}{b+a}$? Hint: Clairaut.

iii) Sketch the several different kinds of behaviors of geodesics on a torus.

<u>Problem</u> (A.6) The surface

$$H(u,v) = {}^{T}[\,u\cos v,\ u\sin v,\ av\,]$$

is a <u>right helicoid</u>; the surface

$$C(u',v') = {}^{T}[\,u'\cos v',\ u'\sin v',\ b\,\mathrm{arcosh}\,\tfrac{u'}{b}\,]$$

is a <u>catenoid</u>

i) Show that the portion $0 \le v < 2\pi$ of a right helicoid is isometric to a catenoid with $b = a$. <u>Hint</u>! Use the fact that the Gauss curvature at corresponding points of the surfaces is the same to discover the relation between u, v and u', v'.

ii) This portion of a right helicoid may be "bent" into a catenoid. Try to express this bending analytically, i.e. by a one-parameter family of isometric surfaces $S(u,v;\alpha)$ $(0 \le \alpha \le 1)$ which gives the helicoid for $\alpha = 0$ and the catenoid for $\alpha = 1$.

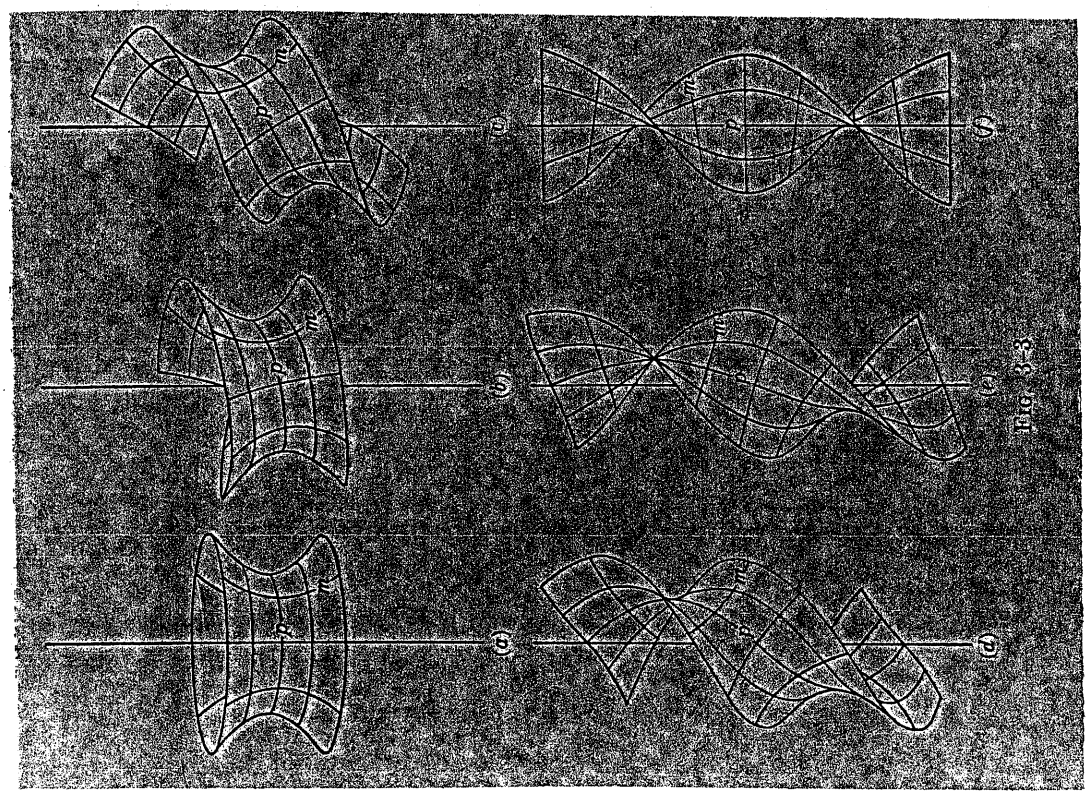


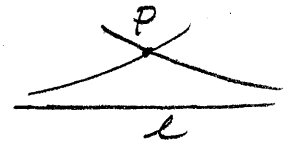
Fig. 3-3

<u>Problem</u> (A.7) The set $\mathbb{R}^2$ equipped with the metric – in geodesic polar coördinates –

$$ds^2 = dr^2 + \left(k \sinh \frac{r}{k}\right)^2 d\theta^2$$

is the "<u>hyperbolic plane</u>". It has constant curvature $K = -\frac{1}{k^2}$. Geodesics may be extended arbitrarily far and are called "lines".

<u>Prove</u> Given a line l and a point P not on l, there are infinitely many lines passing through P and <u>not</u> meeting l.



<u>Remark</u> This is a negation of Euclid's Fifth Postulate. The other postulates <u>are</u> satisfied. The hyperbolic plane is "the" classical non-Euclidean geometry.

<u>Problem</u> (A.8) Show that the hyperbolic plane (Prob. A.7 above) and the Poincaré upper-half-plane – $\mathbb{R}^2$ with $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$, $y > 0$, are <u>isometric</u> by exhibiting explicitly an isometry $(r, \theta) \leftrightarrow (x, y)$. – global.