A. Additional problems

Problem A.1 Let \( \alpha \) be the trefoil knot
\[
\alpha(t) = \begin{bmatrix}
\cos 2t \\
\sin 2t \\
0
\end{bmatrix}(5 + 4 \cos 3t) + \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}4 \sin 3t.
\]
See Problem 2.2.

a) Find the curvature and torsion at the point where the curvature is maximum or minimum. By symmetry it suffices to do one of each, how many are there?

b) Find the (equations of) the osculating plane and circle at the point in part a).

c) Find (numerically) the total curvature of \( \alpha \).

d) Make an accurate sketch.

Problem A.2 a) Let \( \alpha(t) \) be a curve in space and \( \gamma(t) = \alpha(t) + N(t) \frac{1}{x(t)} \) the locus of the centers of curvature.
Show that the tangent to \( \alpha \) at \( \mathbf{x}(t) \) and the tangent to \( \gamma \) at \( \mathbf{y}(t) \) are orthogonal.

b) Illustrate a) with the twisted cube \( x(t) = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} \). Draw carefully an appropriate figure.

Problem A.3 Prove that a closed regular surface in three-dimensional space cannot have Gaussian curvature that is everywhere negative.
Prove that it must have points of positive Gaussian curvature.
Problem 4.4  a) Show that a curve 
\( \mathbf{r}(t) = \mathbf{s}(x(t)) \) on a surface \( S \) is a 
geodesic if and only if 
\[ \mathbf{n}(x(t)) \times \dot{\mathbf{r}}(t) \cdot \dot{\mathbf{r}}(t) = 0, \]
where \( \mathbf{n} \) is the (unit) normal to \( S \)
at \( \mathbf{r}(t) \) and \( ' \dot{\mathbf{r}}/\dot{t}' \) denoted \( d/dt \).

b) Use a) to obtain the geodesics on a 
sphere in Euclidean space.

Problem 4.5  Consider the torus 
\[ S(u,v) = \{ (a + r \cos v) \cos u, (a + r \cos v) \sin u, r \sin v \} \]
where \( 0 < a < b \), in Euclidean space.

i) Express the equations for a geodesic 
other than a parameter curve as 
\[ \frac{du}{dv} = (\text{function of } v) \]

ii) Prove that a geodesic winding once the 
curve \( u = 0 \) at an angle \( \alpha \) \((0 < \alpha < \pi)\)
will also meet the curve \( u = \pi \)
provided that \( \cos \alpha < \frac{b-a}{a} \).

What happens for \( \cos \alpha = \frac{b-a}{a} \) and \( \cos \alpha > \frac{b-a}{a} \) ? Hint: Clairaut.

iii) Sketch the several different kinds 
of behavior of geodesics on a torus,
Problem 4.6. The surface
\[ H(u,v) = \begin{bmatrix} u \cos v, u \sin v, au \end{bmatrix} \]
is a right helicoid; the surface
\[ C(u',v') = \begin{bmatrix} u' \cos v', u' \sin v', b \cosh \frac{u'}{b} \end{bmatrix} \]
is a catenoid.

i) Show that the portion \( 0 \leq v < 2\pi \) of a right helicoid is isometric to a catenoid with \( b = a \). Hint! Use the fact that the Gauss curvature at corresponding points of the surfaces is the same to discuss the relation between \( u,v \) and \( u',v' \).

ii) This portion of a right helicoid may be "sent" into a catenoid. Try to express this binding analytically, i.e. by a one-parameter family of isometric surfaces \( S(u,v;\alpha) \) \( 0 \leq \alpha \leq 1 \) which gives the helicoid for \( \alpha = 0 \) and the catenoid for \( \alpha = 1 \).
Problem 4.7 The set $\mathbb{R}^2$ equipped with
the metric in geodesic polar coordinates
$$\text{d} s^2 = \text{d} r^2 + \left( r \sin \frac{\text{d} \theta}{r} \right)^2 \text{d} \theta^2$$
is the \textit{hyperbolic plane}. It has constant
curvature $K = -\frac{1}{r^2}$. Geodesics may
be extended arbitrarily far and are
called "\textit{lines}.

Prove Given a line $l$ and a point $P$ not on $l$, there
are infinitely many lines
passing through $P$ and not meeting $l$.
Remark: This is a negation of Euclid's
Fifth Postulate. The other postulates
are satisfied. The hyperbolic plane
is "the" classical non-Euclidean
geometry.

Problem 4.8 Show that the hyper-
bolic plane (Prob. 4.7 above) and the
Poincaré upper half-plane
$\mathbb{R}^2$ with
$$\text{d} s^2 = \frac{\text{d} x^2 + \text{d} y^2}{y^2}, \quad y > 0$$
are \textit{isometric} by exhibiting explicitly
an isometry $(r, \theta) \leftrightarrow (x, y)$. - global.