Problem to be turned in: Due 4 June 1971

(a) Let $C(s)$ be a geodesic on an abstract surface and let $C_\varepsilon(s)$ be a differentiable family of geodesics with $C_0 = C$ (a variation of $C$ through geodesics); $s$ denotes arc length on each curve $C_\varepsilon$. Assume that the variation vector field $V(s) = \left( \frac{\partial C_\varepsilon(s)}{\partial \varepsilon} \right)_{\varepsilon=0}$ is normal to $C$ along $C$. (Such variations can be constructed.) Let $N$ be the normal field along $C$ so that $V(s) = \hat{f}(s) N(s)$. Show that $\hat{f}(s)$ satisfies Jacobis equation: $\frac{d^2 \hat{f}}{ds^2} + K(s) \hat{f} = 0$

along $C$, where $K(s)$ denotes Gauss curvature at the point $C(s)$. (Hint: If $C_\varepsilon(s)$ is given by $u^i = u^i(s, \varepsilon)$ in local coordinates, differentiate the equation for geodesics

$$\frac{\partial^2 u^i(s, \varepsilon)}{\partial s^2} + \sum_{j,k=1}^2 \Gamma^i_{jk}(u^i(s, \varepsilon), u^j(s, \varepsilon)) \frac{\partial u^j(s, \varepsilon)}{\partial s} \frac{\partial u^k(s, \varepsilon)}{\partial s} = 0$$

with respect to $\varepsilon$. Express the result in terms of geodesic parallel coordinates (Text, Ch. VII, Sect. 11, §174 ff.) in which the metric is $du^2 + G(u, v) dv^2$, $C$ is described by $u = s, v = 0$, and $V = \frac{\partial}{\partial v}$ along $C$.)

"Text refers to: J. J. STOKER, DIFFERENTIAL GEOMETRY, WILEY-INTERSCIENCE, NEW YORK, 1969."
6) Show that Jacobi's equation has the qualitative interpretation: Nearby geodesics "converge" or "diverge" according as $K > 0$ or $K < 0$. (Compare Jacobi's equation with the equation \( \frac{d^2 \beta}{ds^2} + K_0 \beta = 0 \), $K_0$ = constant, whose solutions can be written down explicitly. Cf. The proof of the Sturm-Liouville Companion Theorem, Text, pp. 228-229.)

Remark: Jacobi's equation is important in the study of geodesics in the large. See Steiner, Ch. VIII, Sect. 12, pp. 247-254.

a) Let $S$ be a closed regular surface in space, $K$ Gauss curvature on $S$. Let \{K \geq 0\} denote the set of points of $S$ where $K \geq 0$. Show that \( \int_{K \geq 0} K dA \geq 4\pi \), where $dA$ is the area element of $S$. (Hint: If $K dA$ is the area element for the spherical image of $S$, it suffices to show that every point of the unit sphere appears at least once as the unit normal to $S$ at a point where $K \geq 0$. Bring an appropriate plane up from infinity.)
(4 continued)

b) Let \( X = X_C(s) \) be a closed regular curve \( C \) in space; \( s \) are lengths along \( C \), \( l \) the length of \( C \) and \( \kappa(s) \), \( N_C(s) \), \( B_C(s) \) the curvature, principal normal, binormal of \( C \) respectively. Contact a tube surface \( S \) around \( C \) by \( X = X(s, \theta) = X_C(s) + \varepsilon (N_C(s) \cos \theta + B_C(s) \sin \theta) \).

Show: For \( \varepsilon > 0 \) small, \( S \) is a closed regular surface; and that the curvature and area element of \( S \) are given by

\[
K = -\frac{X(s) \cos \theta}{\varepsilon (1 - \varepsilon X(s) \cos \theta)} \quad dA = \varepsilon (1 - \varepsilon X(s) \cos \theta) \, ds \, d\theta
\]

respectively. Show that for this surface \( S \);

\[
\int K \, dA = 2 \int \kappa(s) \, ds.
\]

\{K \geq 0\}

c) Combine a) and b) to prove (after K. Voss)

**Fenchel's Theorem:** A closed regular curve in space has total curvature \( \int_0^L \kappa(s) \, ds \geq 2\pi \).