Joint Mathematics Meetings

January 16-19, 1991
San Francisco, California

Wednesday, January 16
AMS Session on Geometry

863-51-706 JAY P. FILLMORE, University of California at San Diego, La Jolla, CA 92093, and ARTHUR SPRINGER, San Diego State University, San Diego, CA 92182, Old and New Euclidean Theorems from Lie Geometry. I. Preliminary report.

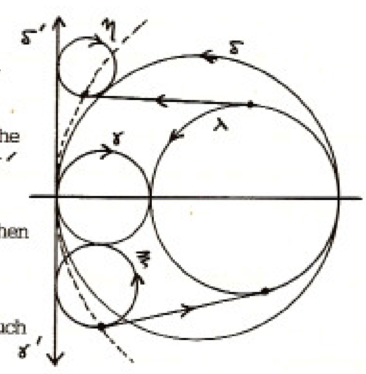
The Lie quadric is the ruled quadric Ω^3 in real projective space P^4. Points of Ω^3 are called cycles, they correspond in the Euclidean plane to: oriented circles, oriented lines, points, and an additional special point. The Apollonius contact problem for cycles has a short elegant solution. Points of P^4 not in Ω^3 are non-cycles. Cycles lying on the polar of a non-cycle are a bunch of cycles. Bunches are described as cycles having fixed relative power (generalizing Steiner power) with respect to a specific cycle, or as cycles having fixed oriented angle with respect to a specific cycle.

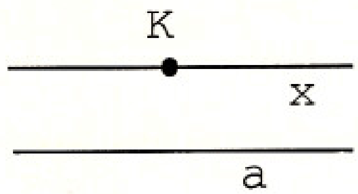
Two cycles touch if the line joining them lies in Ω^3. Such a line is a pencil of touching cycles. That most cycles and non-cycles select a unique cycle from a given pencil serves as the Lie geometry analog of the Fifth Postulate of Euclid. (Received October 10, 1990)

863-51-707 JAY P. FILLMORE, University of California at San Diego, La Jolla, CA 92093, and ARTHUR SPRINGER, San Diego State University, San Diego, CA 92182, Old and New Euclidean Theorems from Lie Geometry. II. Preliminary report.

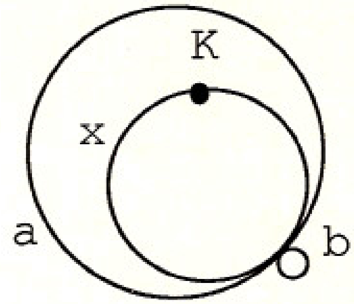
The interpretation of Lie geometry in terms of Euclidean geometry sometimes leads to new and non-trivial theorems in Euclidean geometry.

Theorem. Consider three cycles λ, γ, δ with collinear centers such that λ touches both γ and δ, and γ and δ are tangent as circles but do not touch. The line tangent to the circles γ and δ can be viewed as two oriented lines γ' and δ' touching δ and γ, respectively. Then: If ξ is any cycle touching γ and γ' and if η is any cycle touching δ and δ', then the tangential distance (unoriented) between λ and ξ is the same as that between λ and η. Furthermore, the points of tangency (with respect to the tangential distance) on all such cycles ξ and all such cycles η lie on a circle concentric with λ. (Received October 10, 1990)

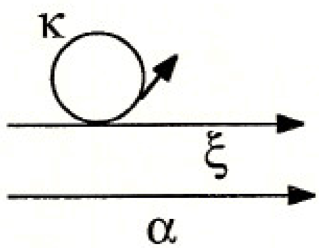




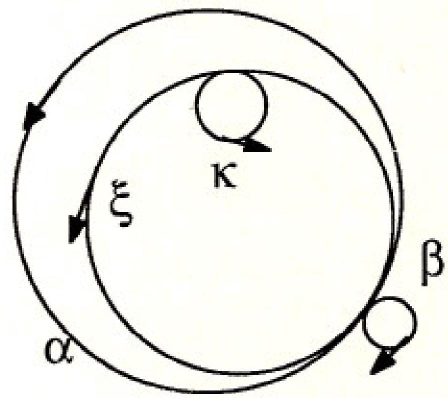
Euclid



Möbius



Laguerre



Lie

$$(x^1 - a^1)^2 + (x^2 - a^2)^2 = (a^r)^2$$

$$(x^1)^2 + (x^2)^2 - 2a^1x^1 - 2a^2x^2 + 2a^s = 0$$

$$(a^1)^2 + (a^2)^2 - (a^r)^2 - 2a^s = 0$$

$$a^i = \alpha^i/\alpha^0 \qquad i = 0,1,2,r,s$$

$$(\alpha^1)^2 + (\alpha^2)^2 - (\alpha^r)^2 - 2\alpha^0\alpha^s = 0$$

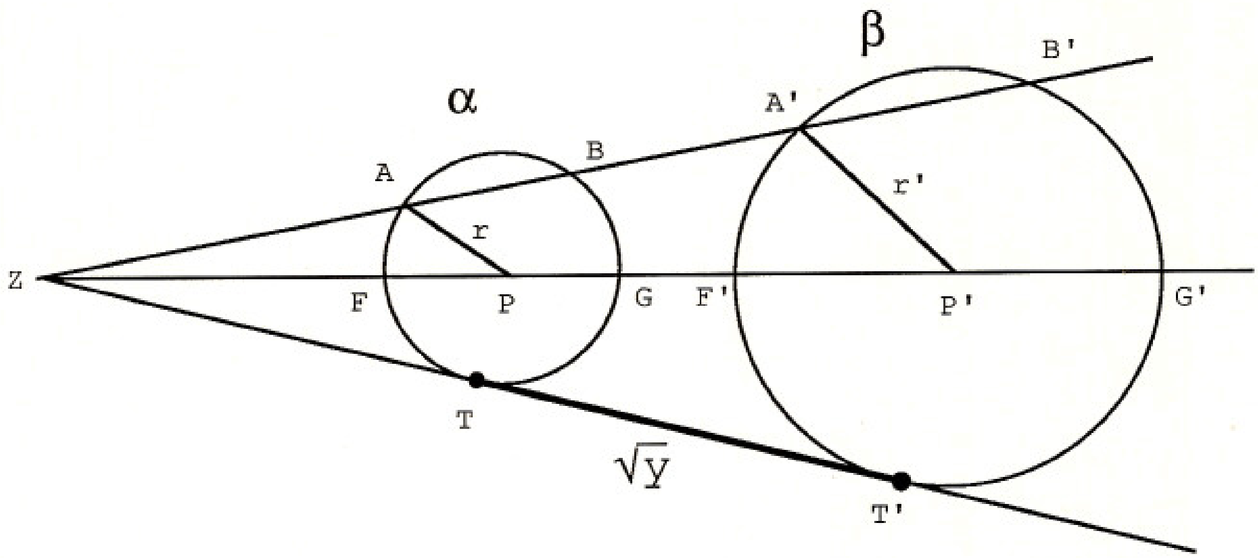
$$(\xi|\eta) = \xi^1\eta^1 + \xi^2\eta^2 - \xi^r\eta^r - \xi^0\eta^s - \xi^s\eta^0$$

Lie cycle Proper cycle

$$\alpha = \begin{pmatrix} \alpha^0 \\ \alpha^1 \\ \alpha^2 \\ \alpha^r \\ \alpha^s \end{pmatrix} \qquad \begin{pmatrix} 1 \\ \text{center} \\ \text{center} \\ \text{radius} \\ \text{Steiner} \end{pmatrix}$$

Lie quadric

$$\Omega^3 = \{ <\xi> \in P^4 \mid (\xi|\xi) = 0 \}$$



Relative power

$$y = AA'.BB' = FF'.GG' = TT'.TT' = PP'.PP' - (r' - r)^2$$

$$\left(-\frac{1}{2}\alpha^0\beta^0\right) y = \alpha^1\beta^1 + \alpha^2\beta^2 - \alpha^r\beta^r - \alpha^0\beta^s - \alpha^s\beta^0$$

$$\left(-\frac{1}{2}\alpha^0\beta^0\right) y = (\alpha|\beta)$$

Touch

cycles $<\alpha>$ and $<\beta>$

$(\alpha|\beta) = 0$

$<\alpha> \subset <\beta>^{\perp}$ and $<\beta> \subset <\alpha>^{\perp}$

$<\alpha, \beta> \subset \Omega^3$

non-cycle $<A>$ and cycle $<\beta>$

$(A|\beta) = 0$

$<\beta> \subset <A>^{\perp}$

Special points

$$\varepsilon_0 = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \quad E_r = \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}, \quad \varepsilon_s = \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$$

proper cycle

touches neither $< \varepsilon_s >$ nor $< E_r >$

$\alpha^0 \neq 0$ and $\alpha^r \neq 0$

point cycle

touches $< E_r >$ but not $< \varepsilon_s >$

$\alpha^r = 0$ and $\alpha^0 \neq 0$

line cycle

touches $< \varepsilon_s >$ but not $< E_r >$

$\alpha^0 = 0$ and $\alpha^r \neq 0$

$\alpha^1 x^1 + \alpha^2 x^2 = \alpha^s$

the special cycle

touches both $< E_r >$ and $< \varepsilon_s >$

$< \varepsilon_s >$ "at infinity"

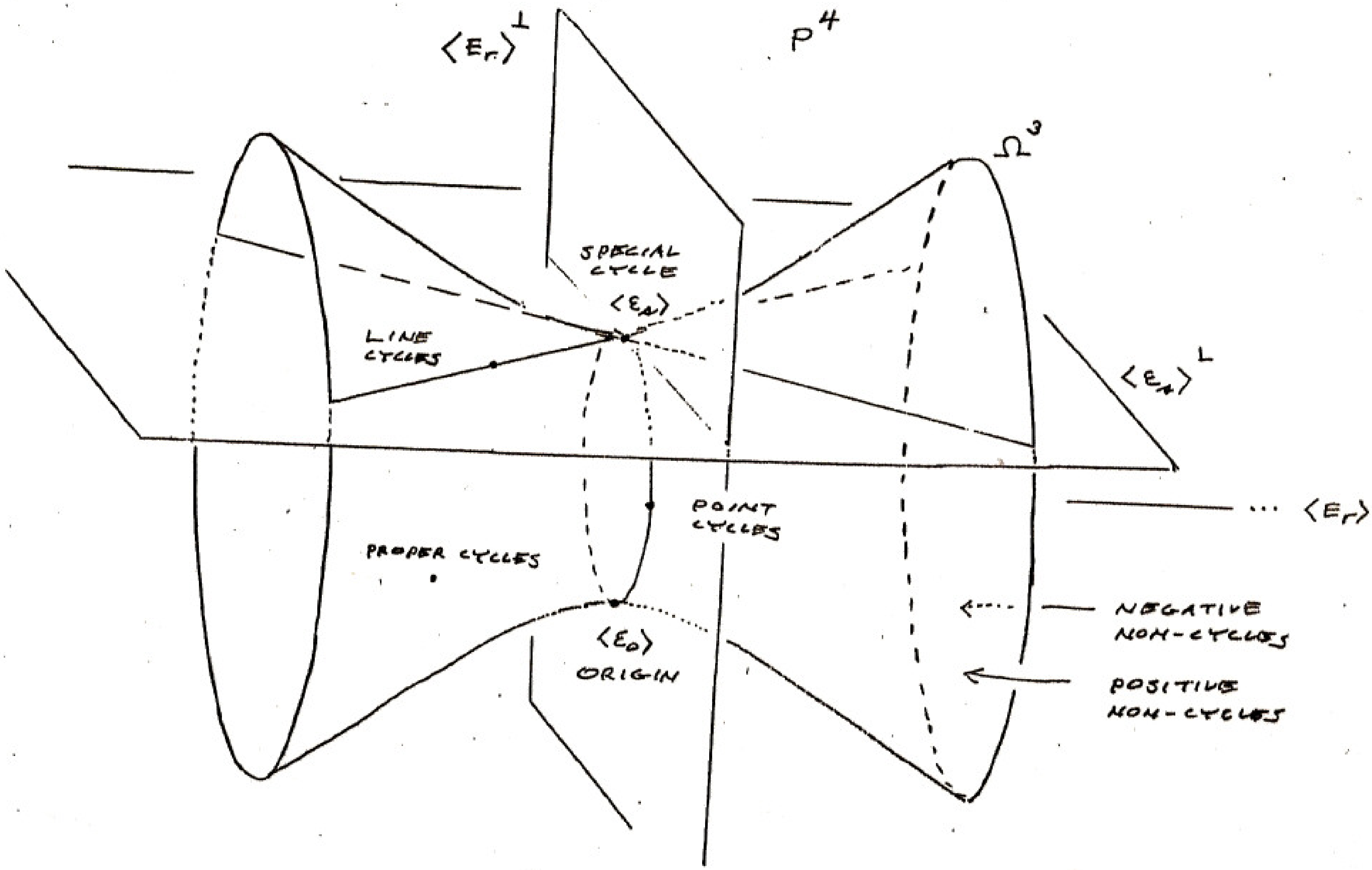


FIGURE 1 - THE LIE QUADRIC

Apollonius contact problem

$< \alpha_1 > \, , \, < \alpha_2 > \, , \, < \alpha_3 >$:

three cycles, no two of which touch

three non-collinear points of Ω^3

all cycles that touch $< \alpha_1 > , < \alpha_2 > , < \alpha_3 >$:

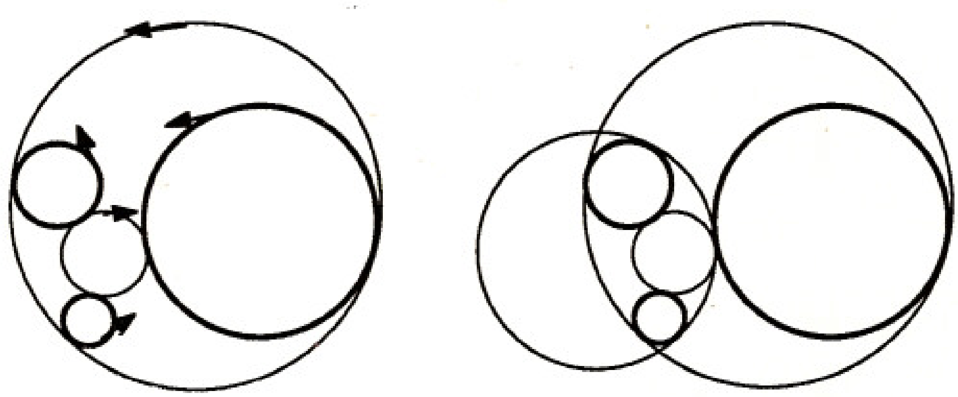
$$(\Omega^3 \cap < \alpha_1 >^{\perp}) \cap (\Omega^3 \cap < \alpha_2 >^{\perp}) \cap (\Omega^3 \cap < \alpha_3 >^{\perp})$$
$$= \Omega^3 \cap < \alpha_1 \, , \, \alpha_2 \, , \, \alpha_3 >^{\perp} = \Omega^3 \cap \text{(line)}$$

$$(\alpha_2 | \alpha_3)\,(\alpha_3 | \alpha_1)\,(\alpha_1 | \alpha_2)\,(E_u | E_u) > 0$$

Calculation of solutions

$$\begin{cases} {}^t\alpha_1 G \xi = 0 \\ {}^t\alpha_2 G \xi = 0 \\ {}^t\alpha_3 G \xi = 0 \end{cases} \qquad G = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$${}^t\xi G \xi = 0 \qquad (\xi | \eta) = \xi^1\eta^1 + \xi^2\eta^2 - \xi^r\eta^r - \xi^0\eta^s - \xi^s\eta^0$$



Apollonius contact problem

Given cycles: $< \alpha_1 >$, $< \alpha_2 >$, $< \alpha_3 >$

Column vectors α_1 , α_2 , α_3 form 5-by-3 matrix $[\ \alpha_1\ \alpha_2\ \alpha_3\ ]$

Augment by any two column vectors κ_4 and κ_5 to obtain

$$K = [\ \alpha_1\ \alpha_2\ \alpha_3\ \kappa_4\ \kappa_5\ ] \qquad \text{5-by-5 } \underline{\text{invertible}} \text{ matrix}$$

Last two columns of

$$({}^t K G)^{-1} = G^{-1}\ {}^t K^{-1} = [\ *\ *\ *\ \xi_4\ \xi_5\ ]$$

are a basis of the solutions to the three linear equations

$${}^t\alpha_1 G \xi = 0\ , \quad {}^t\alpha_2 G \xi = 0\ , \quad {}^t\alpha_3 G \xi = 0$$

For:

$${}^t[\ \alpha_1\ \alpha_2\ \alpha_3\ \kappa_4\ \kappa_5\ ]\ G\ [\ *\ *\ *\ \xi_4\ \xi_5\ ] = {}^t K\ G\ G^{-1}\ {}^t K^{-1} = \text{5-by-5 identity matrix}$$

Quadratic equation

$${}^t(\ \xi_4 + \xi_5\, u\ )\ G\ (\ \xi_4 + \xi_5\, u\ ) = 0 \quad \text{with roots } u \text{ and } u'$$

Set $\beta_1 = \xi_4 + \xi_5\, u \qquad \beta_2 = \xi_4 + \xi_5\, u'$

Solutions $< \beta_1 >$, $< \beta_2 >$: $(\alpha_i | \beta_j) = 0$ for $i = 1,2,3 \quad j = 1,2$

Bunch $\Omega^3 \cap \langle A \rangle^{\perp}$

Laguerre cycle $\langle \lambda \rangle$

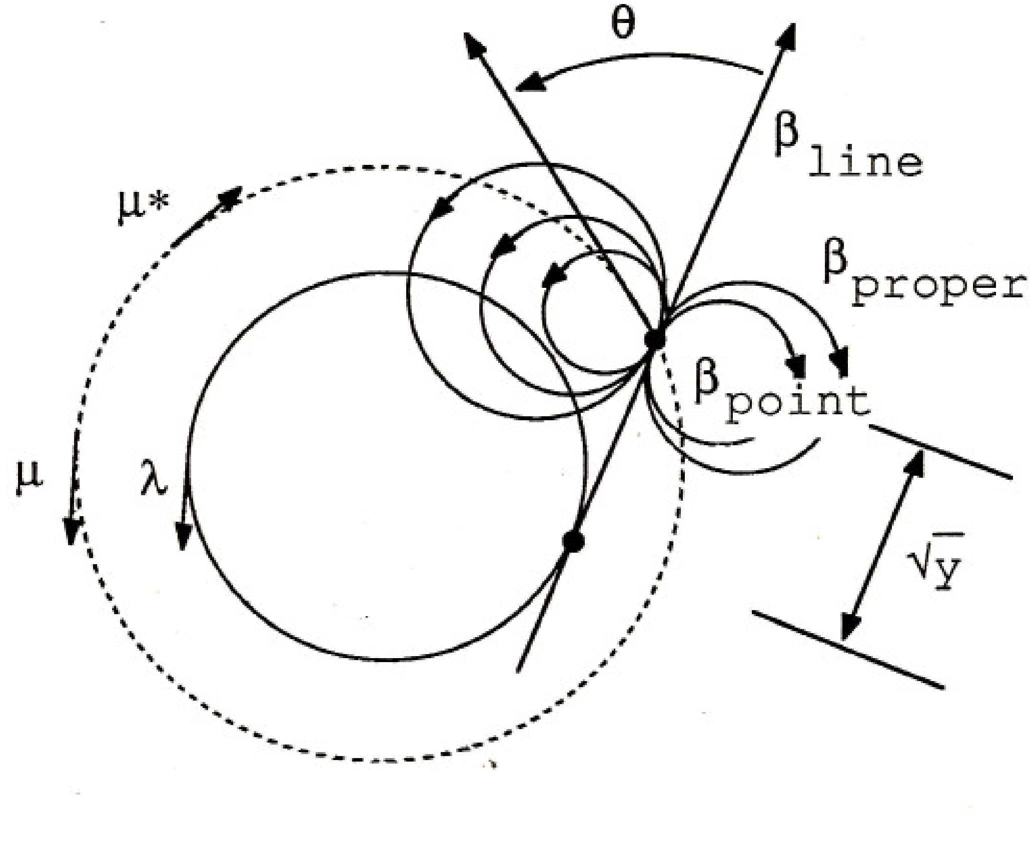
Pole $\langle A \rangle$ not in $\langle \varepsilon_s \rangle^{\perp}$

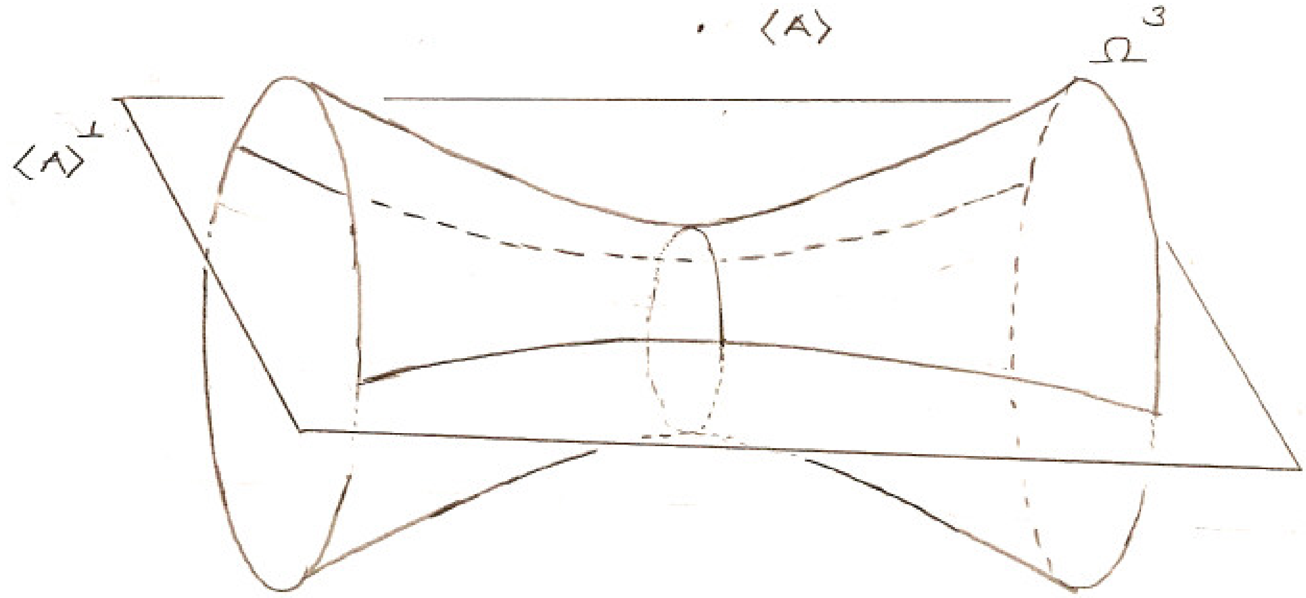
$\Omega^3 \cap \langle A , \varepsilon_s \rangle = \{\langle \varepsilon_s \rangle, \langle \lambda \rangle\}$

Möbius circle $\{\langle \mu \rangle, \langle \mu^* \rangle\}$

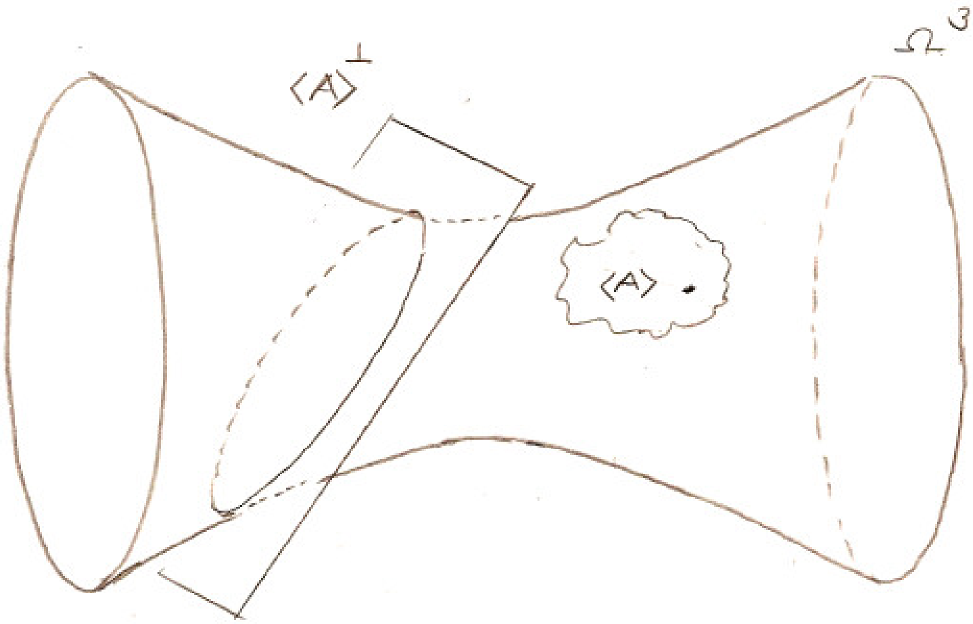
Line $\langle A \rangle$ to $\langle E_r \rangle$ meets Ω^3

$\Omega^3 \cap \langle A , E_r \rangle = \{\langle \mu \rangle, \langle \mu^* \rangle\}$



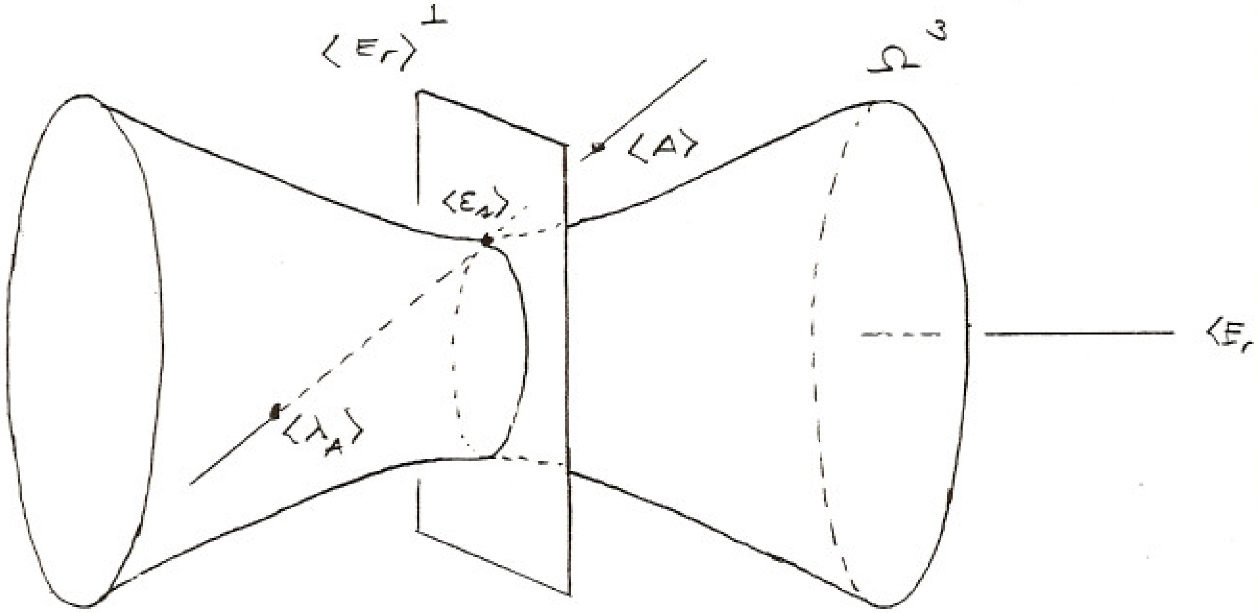


POSITIVE BUNCH - RULED

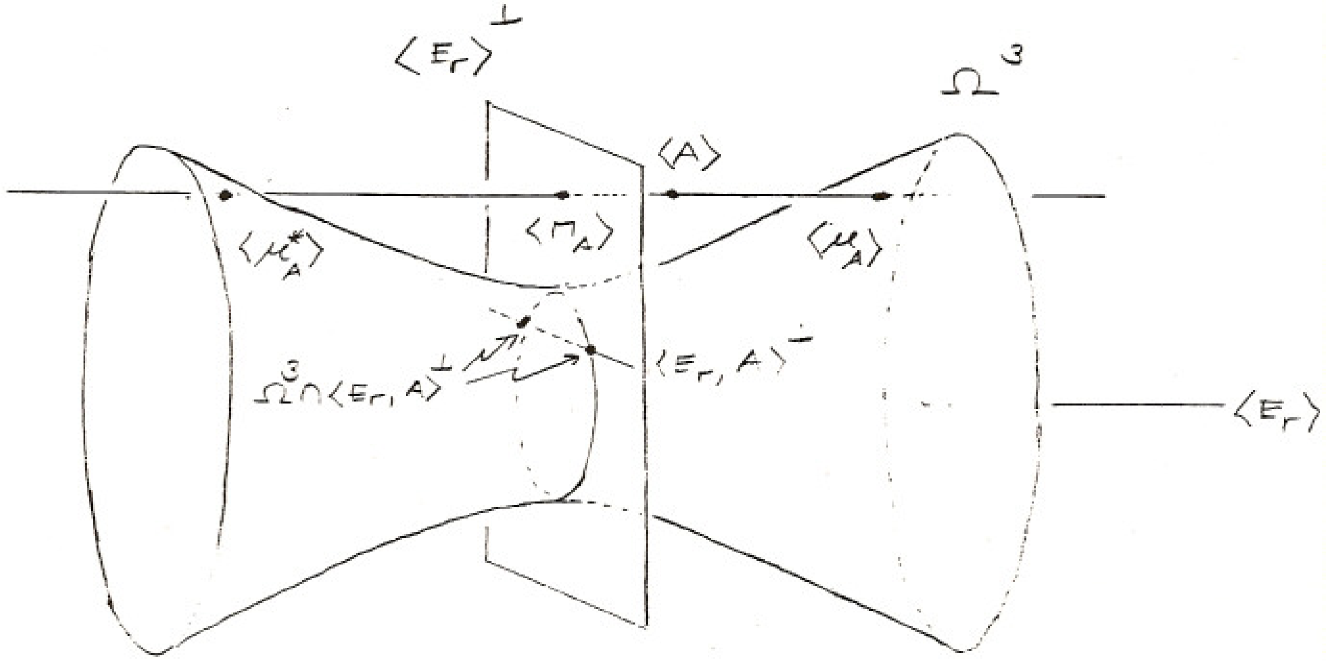


NEGATIVE BUNCH - NON-RULED

FIGURE 2 - BUNCHES



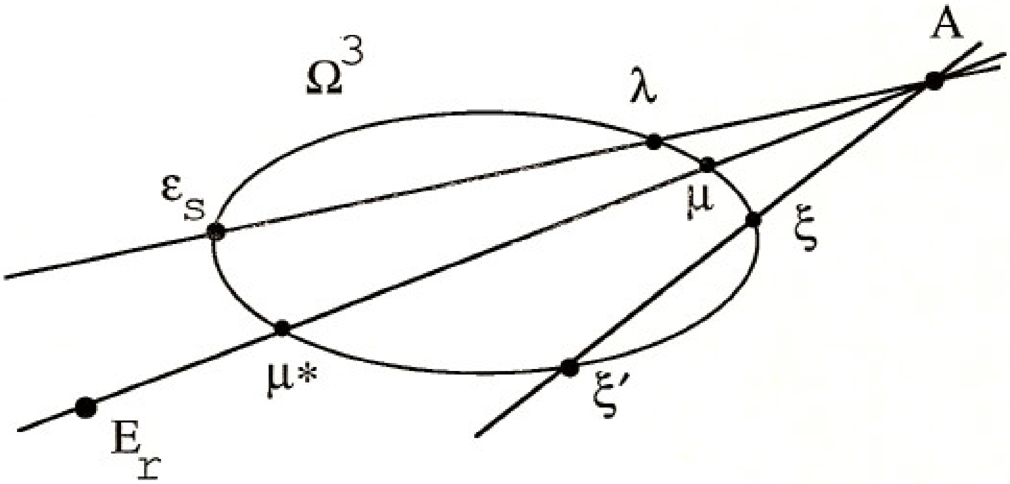
ASSOCIATED LAGUERRE CYCLE



ASSOCIATED MÖBIUS CIRCLE

FIGURE 3

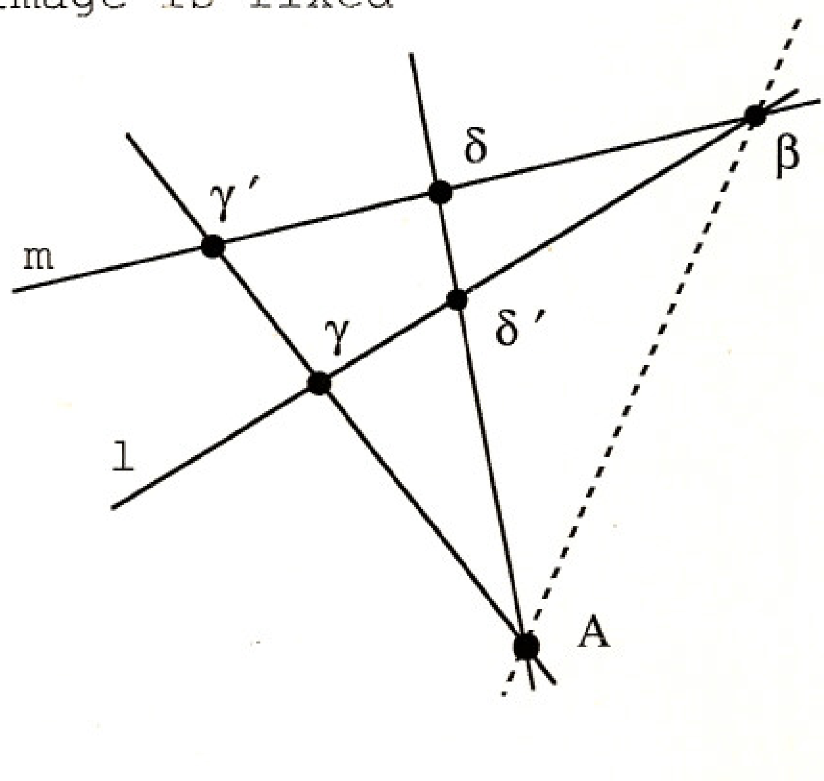
Lie inversion $$\xi' = \xi - A\,2\frac{(A|\xi)}{(A|A)}$$

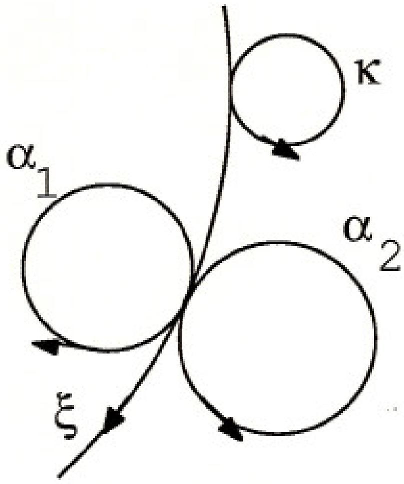


generators of Ω^3 ——> generators of Ω^3

touching cycles ——> touching cycles

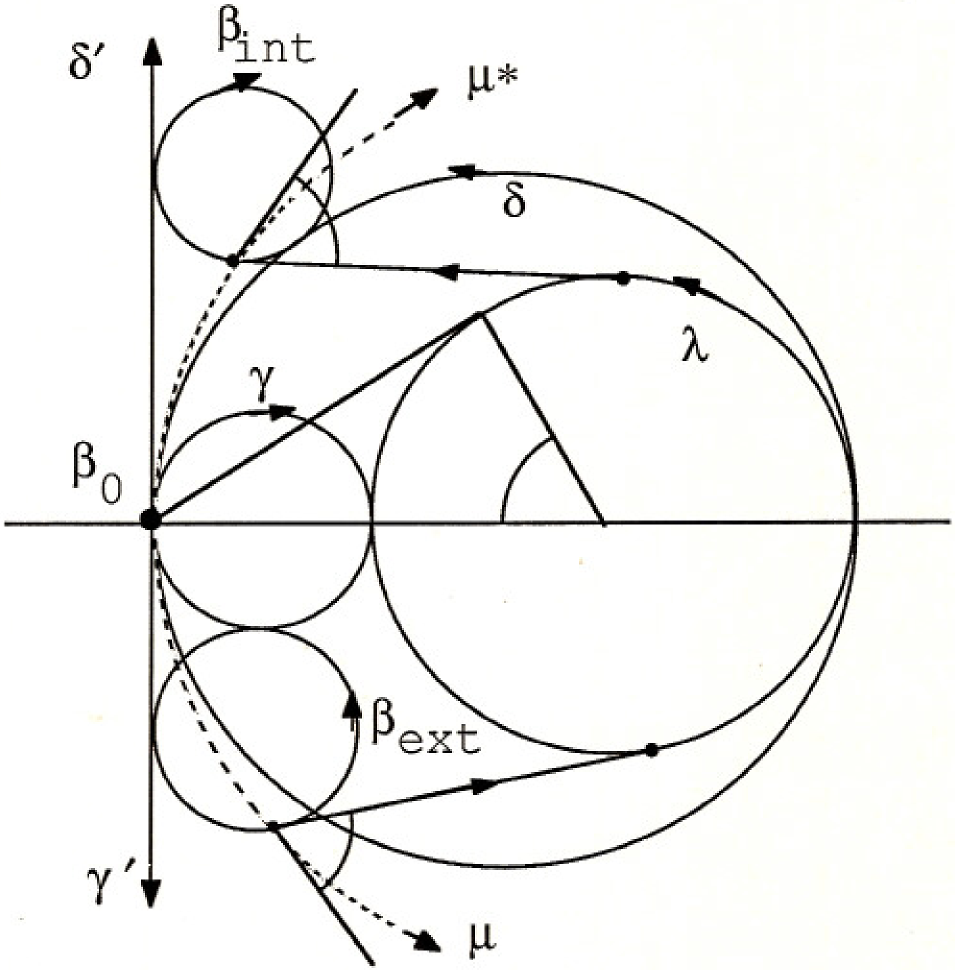
a cycle touching some cycle and its
image is fixed

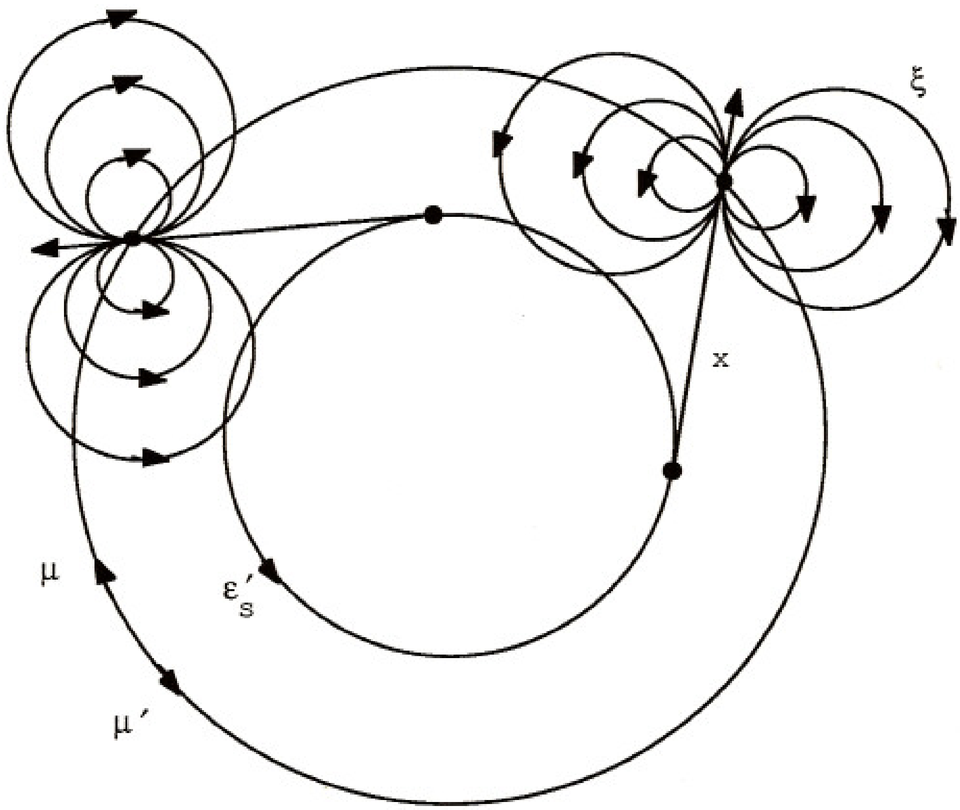




Lemma V

α_1	α_2	κ	ξ
β_0	μ^*	λ	γ
β_0	μ	ε_S	γ'
β_0	μ	λ	δ
β_0	μ^*	ε_S	δ'





ξ
x
μ
ε′s
μ′