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The Generalized Apollonius Contact Problem

Jay P. Fillmore

University of California
at San Diego,
LaJolla, California

Arthur Springer

San Diego State University,

San Diego, California

JAY P. FILLMORE, University of California San Diego, LaJolla, CA 92093
ARTHUR SPRINGER, San Diego State University, San Diego, CA 92182
The Generalized Apollonius Contact Problem. Preliminary report

The classical Apollonius Contact Problem is to determine circles that are tangent to three given circles, or spheres tangent to four given spheres. By generalizing what is a circle, or a sphere, one can solve a host of seemingly unrelated Euclidean constructions.

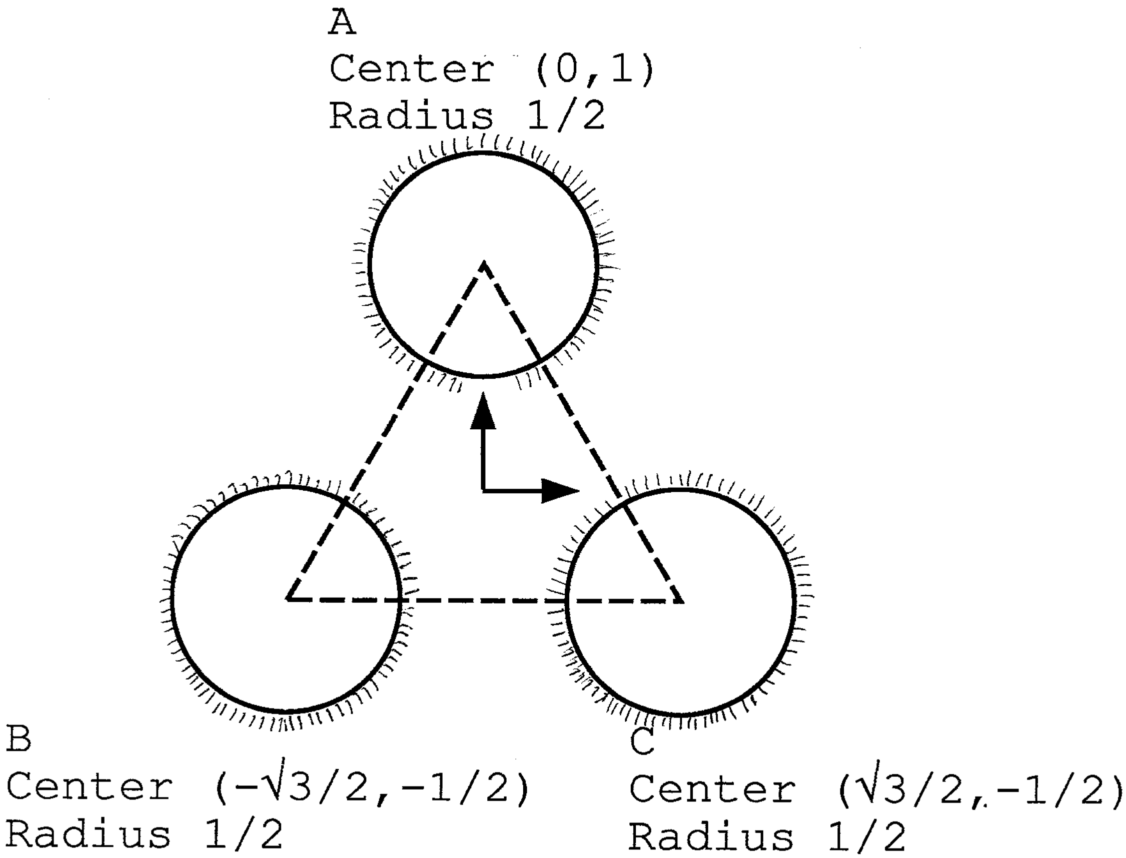
An example. Find all oriented spheres having:

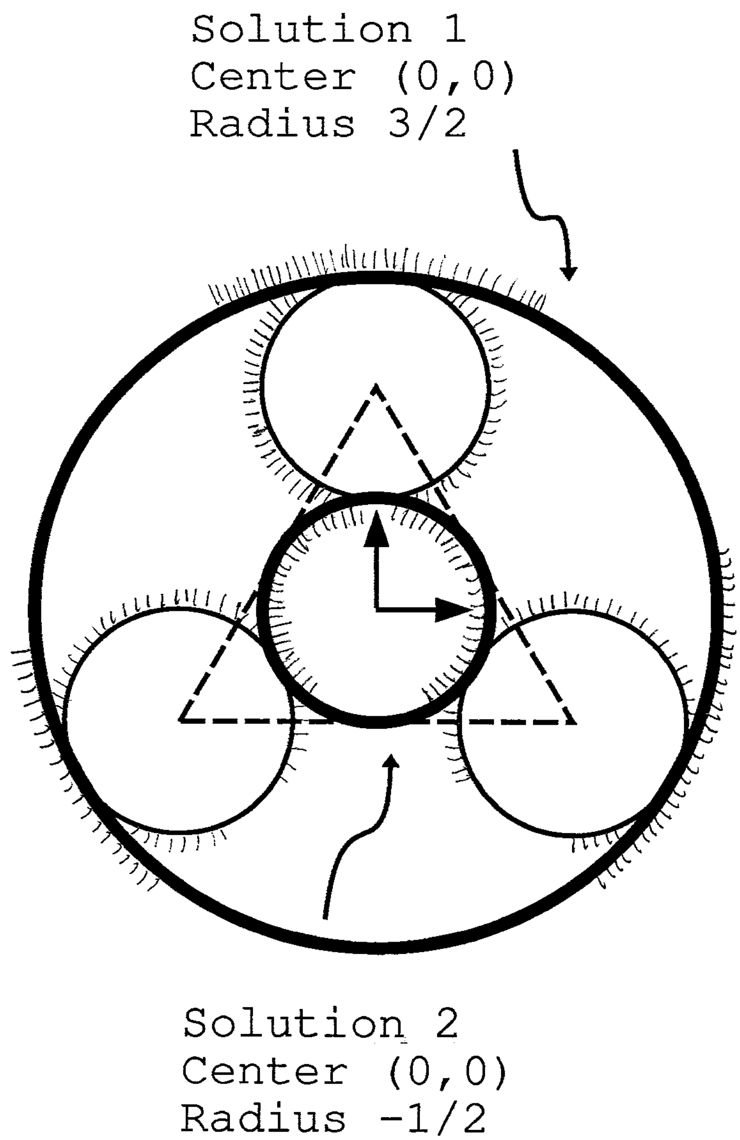
- tangential distance 7 from the oriented sphere: $(x-7)^2 + (y-1)^2 + z^2 = (+2)^2$
- angle $\operatorname{Arccos} \frac{4}{5}$ with the oriented sphere: $(x-5)^2 + y^2 + (z-3)^2 = (-5)^2$
- centers on the plane: $3x + 4y + 12z = 0$
- radius $+1$.

Answer: The two spheres • $x^2 + y^2 + z^2 = (+1)^2$

and • $\left(x - \frac{648}{169}\right)^2 + \left(y + \frac{891}{169}\right)^2 + \left(z - \frac{135}{169}\right)^2 = (+1)^2$

The theory utilizes Lie's Higher Sphere Geometry. The algorithm is easily programmable on a pocket calculator.


A
Center (0,1)
Radius 1/2
B
Center (-√3/2,-1/2)
Radius 1/2
C
Center (√3/2,-1/2)
Radius 1/2


Solution 1
Center (0,0)
Radius 3/2
Solution 2
Center (0,0)
Radius -1/2

Equations of a circle

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 = a_r^2$$

$$x_1^2 + x_2^2 - 2a_1x_1 - 2a_2x_2 + 2a_s = 0$$

Center	(a_1, a_2)
Signed radius	a_r
Steiner power	$2a_s$

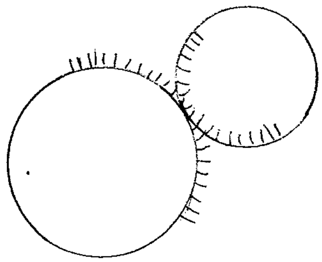
Relation between the parameters

$$a_1^2 + a_2^2 - a_r^2 - 2a_s = 0$$

Representing point in P^4

$$A = {}^t[\,1,\ a_1,\ a_2,\ a_r,\ a_s\,]$$

Two circles touch



$$(b_1 - a_1)^2 + (b_2 - a_2)^2 = (b_r - a_r)^2$$

$$a_1b_1 + a_2b_2 - a_rb_r - a_s - b_s = 0$$

or

$${}^tASA = 0\ ,\ {}^tASB = 0\ ,\ {}^tBSB = 0$$

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

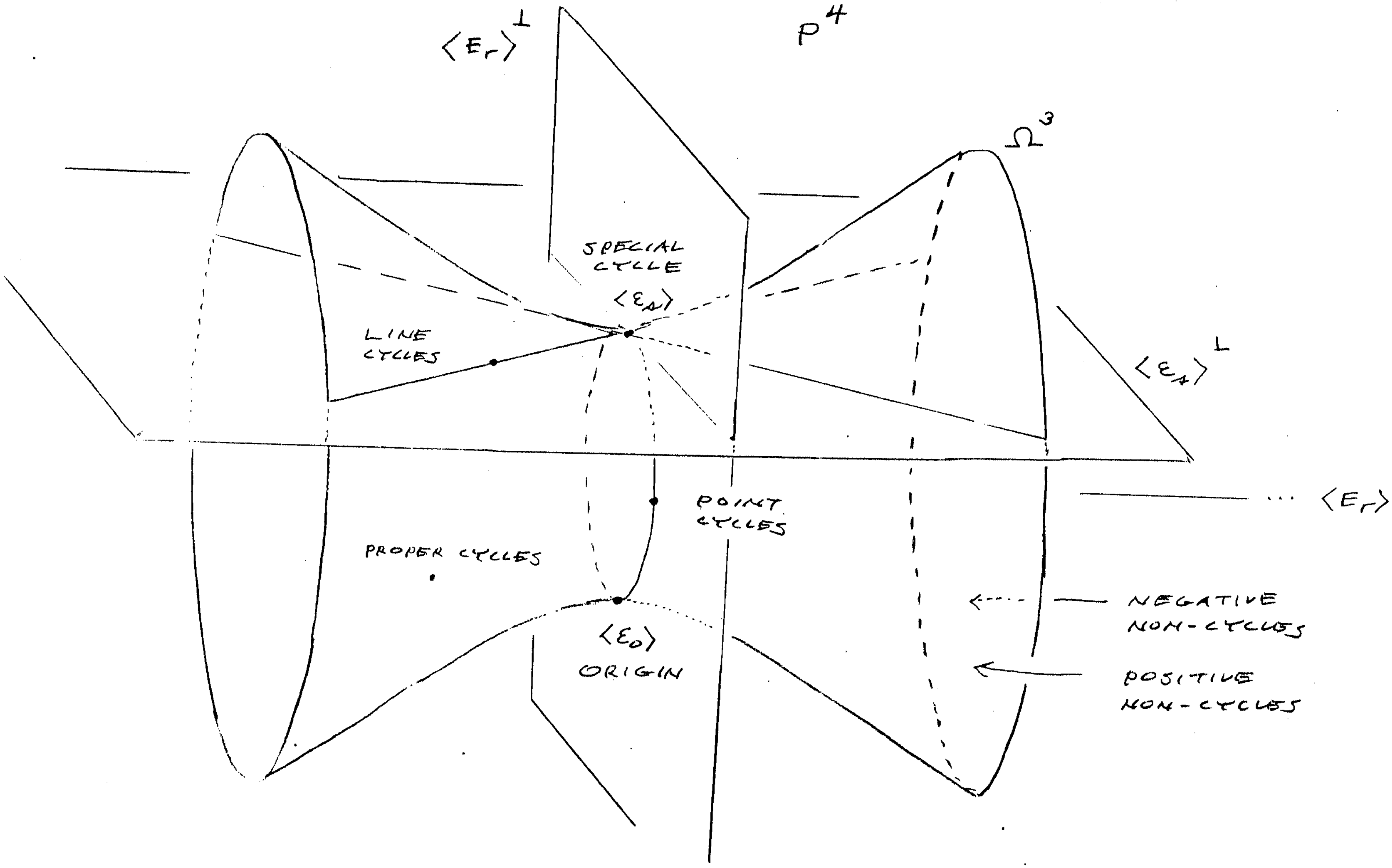


FIGURE 1 - THE LIE QUADRIC

The algorithm in two dimensions

0) Let S be the symmetric matrix

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

.

1) Represent the three conditions by five-dimensional column vectors A , B , C according to the Table.

2) Find any two independent column vectors P and Q so that [a]

$$^{t}ASX = 0, \; ^{t}BSX = 0 \; , \; ^{t}CSX = 0 \; .$$

3) Let u' and u" be the roots of the quadratic equation

$$^{t}(P + Q u) S (P + Q u) = 0 \; .$$

4) The solutions are the oriented circles represented by P + Q u' and P + Q u" .

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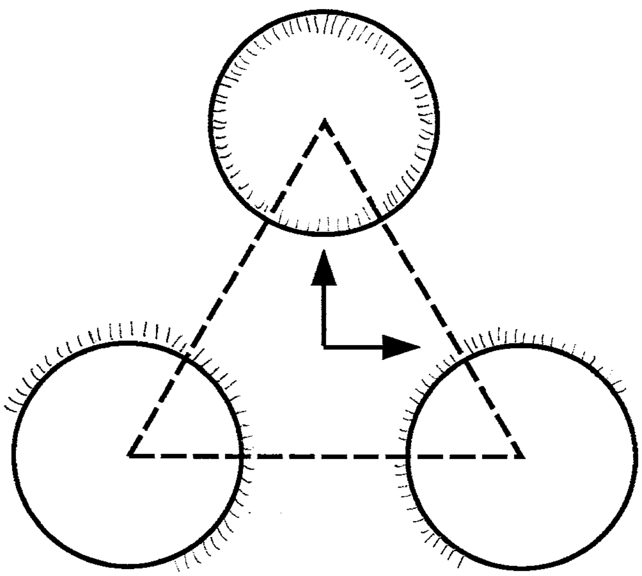
[a] Let K and L be any two columns vectors for which the 5-by-5 matrix [A B C K L] is invertible. Then columns 3 and 4 of $(^{t}[A\ B\ C\ K\ L]S)^{-1}$ will serve for P and Q .

$$A = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1/2 \\ 3/8 \end{bmatrix}$$



$$B = \begin{bmatrix} 1 \\ -\sqrt{3}/2 \\ -1/2 \\ 1/2 \\ 3/8 \end{bmatrix} \qquad C = \begin{bmatrix} 1 \\ \sqrt{3}/2 \\ -1/2 \\ 1/2 \\ 3/8 \end{bmatrix}$$

Example

0) $$S = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

1) Table:

$$A = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1/2 \\ 3/8 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -\sqrt{3}/2 \\ -1/2 \\ 1/2 \\ 3/8 \end{bmatrix} \qquad C = \begin{bmatrix} 1 \\ \sqrt{3}/2 \\ -1/2 \\ 1/2 \\ 3/8 \end{bmatrix}$$

2) Solve $^{t}ASX = 0 \quad ^{t}BSX = 0 \quad ^{t}CSX = 0$:

$$^{t}\begin{bmatrix} 1 & 1 & 1 \\ 0 & -\sqrt{3}/2 & \sqrt{3}/2 \\ 1 & -1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 3/8 & 3/8 & 3/8 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_r \\ x_s \end{bmatrix} = 0$$

$$P = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -3/8 \end{bmatrix} \qquad Q = \begin{bmatrix} 0 \\ 0 \\ -2/3 \\ 1 \\ -1/6 \end{bmatrix}$$

Aside:

$$\begin{bmatrix} -3/8 & 0 & 1 & +1/2 & -1 \\ -3/8 & -\sqrt{3}/2 & -1/2 & -1/2 & -1 \\ -3/8 & \sqrt{3}/2 & -1/2 & -1/2 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} * & * & * & 1 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & 0 & -2/3 \\ * & * & * & 0 & 1 \\ * & * & * & -3/8 & -1/6 \end{bmatrix}$$

3) ${}^t(P + Q u)\, S\, (P + Q u) = 0$:

$\frac{3}{4} + \frac{2}{6} u - \frac{5}{9} u^2 = 0$ has roots $\frac{3}{2}$ and $-\frac{9}{10}$.

4) Solutions:

$$P + Q.\left(-\frac{9}{10}\right) = \begin{bmatrix} 1 \\ 0 \\ 3/5 \\ -9/10 \\ -9/40 \end{bmatrix} \quad \text{and} \quad P + Q.\frac{3}{2} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3/2 \\ -5/8 \end{bmatrix}$$

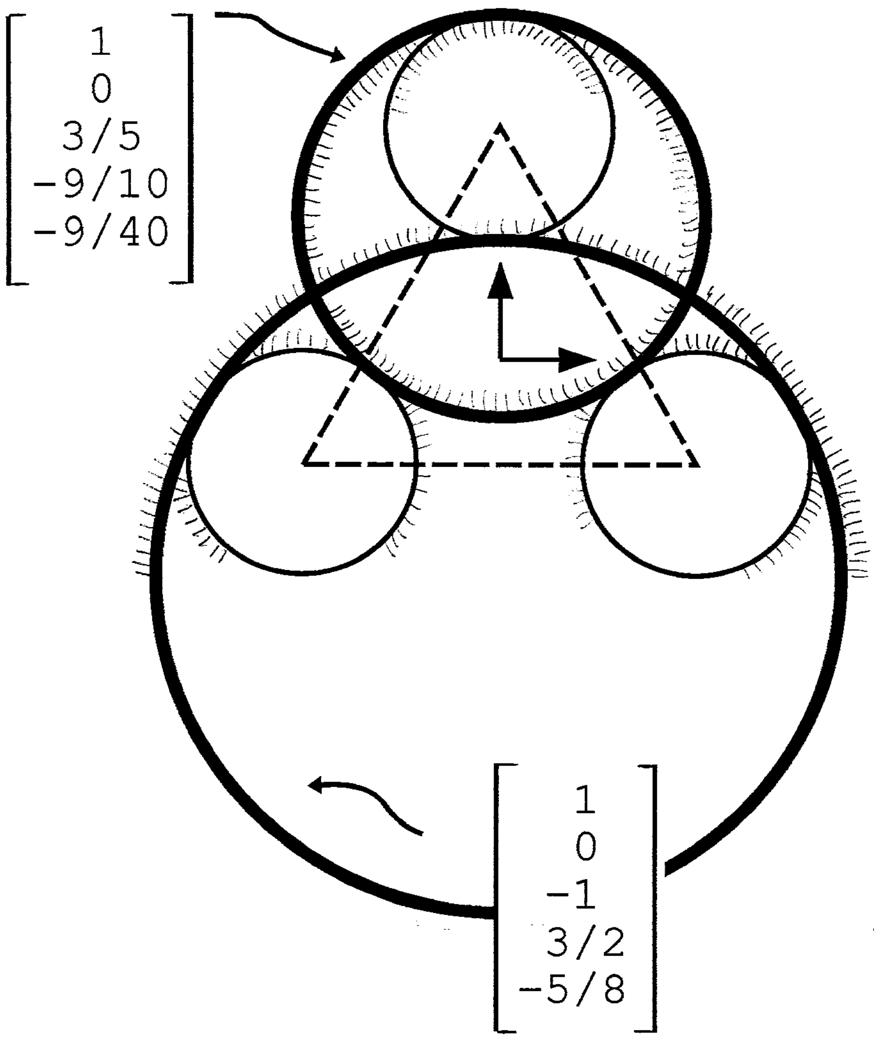

[1 0 3/5 −9/10 −9/40]
[1 0 −1 3/2 −5/8]

Table - 2D

Condition	Column vector
Touch circle with center (a_1,a_2,a_3) and radius a_3 CLASSICAL CONDITION	${}^t[\ 1\ ,\ a_1\ ,\ a_2\ ,\ a_r\ ,\ a_s\ ]$
Tangential distance (oriented) to circle or point is d	${}^t[\ 1,\ a_1,\ a_2,\ a_r,\ a_s - \frac{1}{2} d^2\ ]$
Relative power with respect to circle or point is y	${}^t[\ 1,\ a_1,\ a_2,\ a_r,\ a_s - \frac{1}{2} y\ ]$
Signed angle with respect to orinted circle is θ	${}^t[\ 1,\ a_1,\ a_2,\ a_r \cos \theta,\ a_s\ ]$
Signed angle with respect to plane is θ	${}^t[\ 0,\ u_1,\ u_2,\ \cos \theta,\ u_s\ ]$
Touch plane with unit normal (u_1,u_2,u_3)	${}^t[\ 0\ ,\ u_1\ ,\ u_2\ ,\ 1\ ,\ u_s\ ]$
Center on plane with unit normal (u_1,u_2,u_3)	${}^t[\ 0,\ u_1\ ,\ u_2\ ,\ 0\ ,\ u_s\ ]$
Signed radiis is R	${}^t[\ 0\ ,\ 0\ ,\ 0\ ,\ 1\ ,\ -R\ ]$

<u>Example</u>. To find all oriented circles having:

- tangential distance 7 from the oriented circle $(x_1 - 7)^2 + (x_2 - 1)^2 = (+2)^2$

- angle $\text{Arccos}\,\frac{4}{5} = 36°52'$ with the oriented circle $(x_1 - 5)^2 + (x_2 - 3)^2 = (-5)^2$

- centers on the plane: $5x_1 + 12x_2 = 0$

1) From table:

$$A = \begin{bmatrix} 1 \\ 7 \\ 1 \\ 2 \\ 23 - \frac{1}{2}7^2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 5 \\ 3 \\ -5\cdot\frac{4}{5} \\ \frac{9}{2} \end{bmatrix} \qquad C = \begin{bmatrix} 0 \\ \frac{5}{13} \\ \frac{12}{13} \\ 0 \\ 0 \end{bmatrix}$$

2) Solve $^tASX = 0 \quad ^tBSX = 0 \quad ^tCSX = 0$:

$$^t\begin{bmatrix} 2 & 2 & 0 \\ 14 & 10 & 5 \\ 2 & 6 & 12 \\ 4 & -8 & 0 \\ -3 & 9 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_r \\ x_s \end{bmatrix} = 0$$

$$P = \begin{bmatrix} 406 \\ 36 \\ -15 \\ 423 \\ 0 \end{bmatrix} \qquad Q = \begin{bmatrix} -34 \\ 72 \\ -30 \\ 0 \\ 423 \end{bmatrix}$$

3) $^t(P + Q u) S (P + Q u) = 0$:

$$(-177408) + 2(-168696) u + 34848 u^2 = 0$$

has roots $\frac{112}{11}$ and $-\frac{1}{2}$.

4) Solutions:

$$P.11 + Q.112 = \begin{bmatrix} 14 \\ 180 \\ -75 \\ 99 \\ 1008 \end{bmatrix}.47$$

and

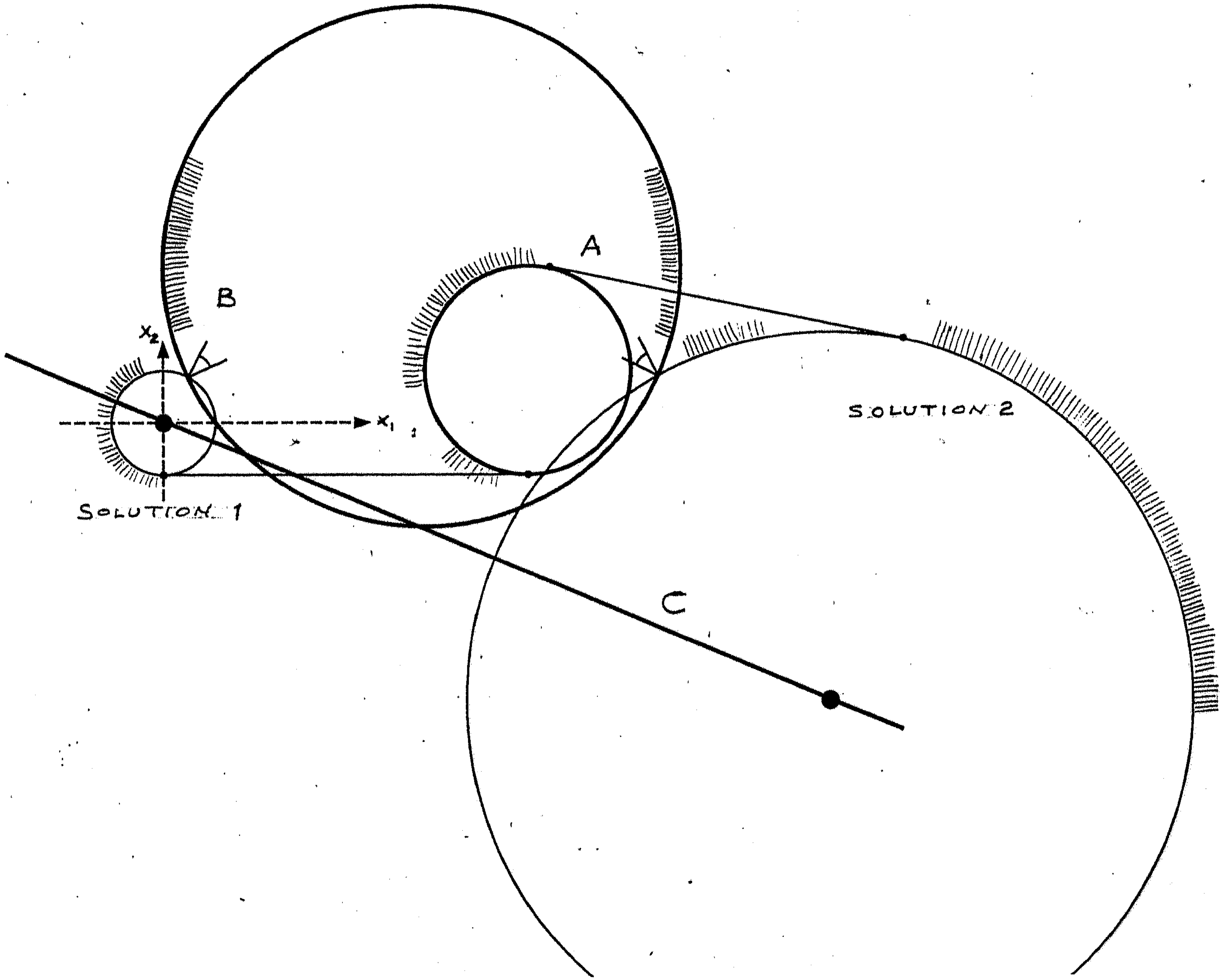
$$P.2 + Q.(-1) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -\ 1/2 \end{bmatrix}.846$$

<u>Answer</u>: The two circles

- $x_1{}^2 + x_2{}^2 = (+1)^2$

and

- $\left(x_1 - \frac{90}{7}\right)^2 + \left(x_2 + \frac{75}{14}\right)^2 = \left(+ \frac{99}{14}\right)^2$


A
B
x_2
x_1
SOLUTION 1
SOLUTION 2
C

The algorithm - 3D

0) Let S be the symmetric matrix

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

1) Represent the four conditions by six-dimensional column vectors A , B , C , D according to the Table.

2) Find any two independent column vectors P and Q so that [a)]

$${}^tASX = 0, \; {}^tBSX = 0 , \; {}^tCSX = 0 , \; {}^tDSX = 0 .$$

3) Let u' and u" be the roots of the quadratic equation

$${}^t(P + Q u) S (P + Q u) = 0 .$$

4) The solutions are the oriented spheres represented by P + Q u' and P + Q u" .

[a)] Let K and L be any two columns vectors for which the 6-by-6 matrix [A B C D K L] is invertible. Then columns 5 and 6 of $({}^t[A\ B\ C\ D\ K\ L]S)^{-1}$ will serve for P and Q .

Table - 3D

Condition	Column vector
Touch sphere with center (a_1,a_2,a_3) and radius a_3 CLASSICAL CONDITION	${}^t[\ 1\ ,\ a_1\ ,\ a_2\ ,\ a_3\ ,\ a_r\ ,\ a_s\ ]$
Tangential distance (oriented) to sphere or point is d	${}^t[\ 1,\ a_1,\ a_2,\ a_3,\ a_r,\ a_s - \frac{1}{2} d^2\ ]$
Relative power with respect to sphere or point is y	${}^t[\ 1,\ a_1,\ a_2,\ a_3,\ a_r,\ a_s - \frac{1}{2} y\ ]$
Signed angle with respect to orinted sphere is θ	${}^t[\ 1,\ a_1,\ a_2,\ a_3,\ a_r \cos\theta\ ,\ a_s\ ]$
Signed angle with respect to plane is θ	${}^t[\ 0,\ u_1,\ u_2,\ u_3,\ \cos\theta\ ,\ u_s\ ]$
Touch plane with unit normal (u_1,u_2,u_3)	${}^t[\ 0\ ,\ u_1\ ,\ u_2\ ,\ u_3\ ,\ 1\ ,\ u_s\ ]$
Center on plane with unit normal (u_1,u_2,u_3)	${}^t[\ 0,\ u_1,\ u_2,\ u_3,\ 0\ ,\ u_s\ ]$
Signed radiis is R	${}^t[\ 0\ ,\ 0,\ 0,\ 0,\ 1\ ,\ -R\ ]$

Equations of a sphere

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 = a_r^2$$

$$x_1^2 + x_2^2 + x_3^2 - 2a_1x_1 - 2a_2x_2 - 2a_2x_3 + 2a_s = 0$$

Center (a_1, a_2, a_3)

Signed radius a_r

Steiner power $2a_s$

Relation between the parameters

$$a_1^2 + a_2^2 + a_3^2 - a_r^2 - 2a_s = 0$$

Representing point in P^5

$$A = {}^t[1, a_1, a_2, a_3, a_r, a_s]$$

Two spheres touch

$$(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2 = (b_r - a_r)^2$$

$$a_1b_1 + a_2b_2 + a_3b_3 - a_rb_r - a_s - b_s = 0$$

or

$${}^tASA = 0 \ , \ {}^tASB = 0 \ , \ {}^tBSB = 0$$

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>An example</u>. Find all oriented spheres having:

- tangential distance 7 from the oriented sphere
  $$(x_1 - 7)^2 + (x_2 - 1)^2 + x_3^2 = (+2)^2$$
- angle $\text{Arccos}\ \frac{4}{5}$ with the oriented sphere
  $$(x_1 - 5)^2 + x_2^2 + (x_3 - 3)^2 = (-5)^2$$
- centers on the plane
  $$3x_1 + 4x_2 + 12x_3 = 0$$
- radius $+1$.

1) From table:

$$A = \begin{bmatrix} 1 \\ 7 \\ 1 \\ 0 \\ 2 \\ 23 - \frac{1}{2}\,7^2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 3 \\ -5\cdot\frac{4}{5} \\ \frac{9}{2} \end{bmatrix} \qquad C = \begin{bmatrix} 0 \\ \frac{3}{13} \\ \frac{4}{13} \\ \frac{12}{13} \\ 0 \\ 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

2) Solve ${}^{t}ASX = 0 \quad {}^{t}BSX = 0 \quad {}^{t}CSX = 0 \quad {}^{t}DSX = 0$:

$$
{}^{t}\begin{bmatrix} 2 & 2 & 0 & 0 \\ 14 & 10 & 3 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 6 & 12 & 0 \\ 4 & -8 & 0 & 1 \\ -3 & 9 & 0 & -1 \end{bmatrix}
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_r \\ x_s \end{bmatrix} = 0
$$

$$
P = \begin{bmatrix} 24 \\ 24 \\ -33 \\ 5 \\ 24 \\ 123 \end{bmatrix} \qquad
Q = \begin{bmatrix} 270 \\ 24 \\ -33 \\ 5 \\ 270 \\ 0 \end{bmatrix}
$$

3) ${}^{t}(P + Q\,u)\,S\,(P + Q\,u) = 0$:

$$(-4790) + 2(-38000)\,u + (-71210)\,u^2 = 0$$

has roots -1 and $-\dfrac{479}{7121}$.

4) Solutions:

$$P + Q.(-1) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1/2 \end{bmatrix} .(-246) \quad \text{and} \quad P.2 + Q.\left(-\frac{479}{7121}\right) = \begin{bmatrix} 1 \\ \frac{648}{169} \\ -\frac{891}{169} \\ \frac{135}{169} \\ 1 \\ \frac{7121}{338} \end{bmatrix} . \frac{338 \times 123}{7121}$$

<u>Answer</u>: The two spheres

- $x_1^2 + x_2^2 + x_3^2 = (+1)^2$

and

- $\left(x_1 - \frac{648}{169}\right)^2 + \left(x_2 + \frac{891}{169}\right)^2 + \left(x_3 - \frac{135}{169}\right)^2 = (+1)^2$