The Lie quadric

Standing assumptions:

k commutative field of char $\neq 2$

V n-dimensional vector space over k, with elements $\vec{x}, \vec{y}, \ldots$

$\vec{x}\cdot\vec{x}$ non-singular quadratic form with polarization $\vec{x}\cdot\vec{y}$.

Main example: Euclidean space

$k = \mathbb{R}$

$V = \mathbb{R}^n$ (col.) consisting of column vectors $\vec{x} = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$

$$\vec{x}\cdot\vec{y} = {}^t\!\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix} = x^1 y^1 + \cdots + x^n y^n$$

where t denotes transpose

and $A = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = \begin{pmatrix} \text{Gram matrix of} \\ \text{the quad. form} \end{pmatrix}$.

Equation of a hypersphere:

$$(\vec{x} - \vec{a})\cdot(\vec{x} - \vec{a}) = (a^r)^2$$

where

$\vec{a} \in V$ is the <u>center</u>

$a^r \in k$ is the <u>radius</u>

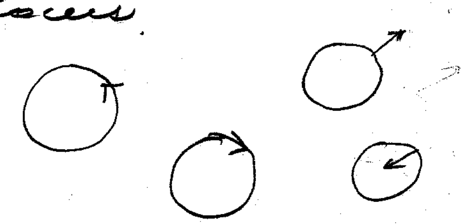
A *hypersphere* is the locus of points $\vec{x} \in V$ satisfying such an equation. It may be empty and one then has only the equation.

Fixed points of inversion
$$\vec{x}' = \vec{a} + (\vec{x} - \vec{a})\frac{(a^r)^2}{(\vec{x}-\vec{a})\cdot(\vec{x}-\vec{a})}$$

Note! The radii a^r and $-a^r$ yield the same equation. We will say that they give opposite *orientations* to the hypersphere which is the locus.

Eg $n = 2$

$$x_1^2 + x_2^2 = 1^2$$
$$x_1^2 + x_2^2 = (-1)^2$$



Rewrite equation of hypersphere as
$$\vec{x}\cdot\vec{x} - 2\,\vec{a}\cdot\vec{x} + a^s = 0$$
where
$$a^s = \vec{a}\cdot\vec{a} - (a^r)^2 = \left(\begin{array}{l}\text{Steiner power}\\ \text{of } \vec{0} \text{ in } V \text{ with}\\ \text{respect to the}\\ \text{hypersphere}\end{array}\right)$$

Then! $\vec{a}$, a^r, a^s are a *redundant* set of coordinates specifying an oriented hypersphere.

We may include *hyperplanes* of V by making *these* coordinates *homogeneous*:

α^0 in k scale factor for homog. coords.

$$\left.\begin{array}{ll} \vec{a} = \vec{\alpha}/\alpha^0 & \text{in } V \\ a^r = \alpha^r/\alpha^0 & \text{in } k \\ a^s = \alpha^s/\alpha^0 & \text{in } k \end{array}\right\} \text{ when } \alpha^0 \neq 0$$

and relation of redundancy becomes

$$\frac{\alpha^s}{\alpha^0} = \frac{\vec{\alpha}}{\alpha^0} \cdot \frac{\vec{\alpha}}{\alpha^0} - \left(\frac{\alpha^r}{\alpha^0}\right)^2$$

or $$\boxed{\vec{\alpha} \cdot \vec{\alpha} - (\alpha^r)^2 - \alpha^0 \alpha^s = 0}$$

Thus, the equation is

$$\alpha^0\, \vec{x} \cdot \vec{x} - 2\vec{\alpha} \cdot \vec{x} + \alpha^s = 0$$

with orientation α^r

related by $\square$.

1) $\alpha^0 \neq 0$

$$\left(\vec{x} - \frac{\vec{\alpha}}{\alpha^0}\right) \cdot \left(\vec{x} - \frac{\vec{\alpha}}{\alpha^0}\right) = \frac{\alpha^s}{\alpha^0} - \frac{\vec{\alpha} \cdot \vec{\alpha}}{(\alpha^0)^2} = \left(\frac{\alpha^r}{\alpha^0}\right)^2,$$

is equation of a hypersphere with

$\frac{\vec{\alpha}}{\alpha^0}$ = center

$\frac{\alpha^r}{\alpha^0}$ = radius, signed for orientation

$\frac{\alpha^s}{\alpha^0}$ = Steiner power

The condition for a hypersphere (non-empty locus) is:

$\vec{\alpha} \cdot \vec{\alpha} - \alpha^0 \alpha^s$ is a square in k

Eg In Euclidean case, this is ≥ 0.

Note: Independently we have:

i) $\alpha^{\Gamma} \neq 0$ non-singular hypersphere

$\alpha^{\Gamma} = 0$ singular hypersphere

(point sphere in Euclidean case)

and

ii) $\alpha^{\perp} \neq 0$ not passing through origin

$\alpha^{\perp} = 0$ passing through origin

2) $\alpha^{\circ} = 0$

$$\vec{\alpha} \cdot \vec{x} = \frac{\alpha^{\perp}}{2}$$ is equation of a hyperplane with normal vector $\vec{\alpha}$.

i) $\vec{\alpha} \neq 0$ $\quad$ $\vec{\alpha} \cdot \vec{\alpha} \neq 0$ non-singular hyperplane

$\vec{\alpha} \cdot \vec{\alpha} = 0$ singular hyperplane

ii) $\vec{\alpha} = 0$ $\quad$ Then $\alpha^{\Gamma} = 0$ by $\square$, so this is the special case

$\alpha^{\circ} = 0$, $\vec{\alpha} = 0$, $\alpha^{\Gamma} = 0$, $\alpha^{\perp} \neq 0$

to be interpreted later.