Stiefel manifold

4) E^n = Euclidean n-space

$$\vec{x}\cdot\vec{x} = \sum_{i=1}^{n} (x^i)^2$$

$$(\xi|\eta) = {}^t\xi L \eta, \qquad L = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 1_n & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_0 = 1$$

a). The Möbius quadric.

$$\Psi^n = \langle e_r \rangle^\perp \cap \Omega^{n+1}$$

In $\langle e_r\rangle^\perp = \{ e_0\xi^0 + \vec\xi + e_\infty \xi^\infty \}$, $\left\{ \vec\xi\cdot\vec\xi - \xi^0\xi^\infty = 0 \right.$

we have $\xi^0 + \xi^\infty \neq 0$.

For: If $\xi^0 + \xi^\infty = 0$, then

$$\vec\xi\cdot\vec\xi + (\xi^0)^2 = \vec\xi\cdot\vec\xi - \xi^0\xi^\infty = 0$$

gives $\vec\xi = 0$ & $\xi^0 = 0$, so $\xi^\infty = 0$ —

Now write $\vec\xi\cdot\vec\xi - \xi^0\xi^\infty = 0$

as

$$\vec\xi\cdot\vec\xi + \left(\frac{\xi^0-\xi^\infty}{2}\right)^2 = \left(\frac{\xi^0+\xi^\infty}{2}\right)^2$$

or

$$\frac{2\vec\xi}{\xi^0+\xi^\infty}\cdot\frac{2\vec\xi}{\xi^0+\xi^\infty} + \left(\frac{\xi^0-\xi^\infty}{\xi^0+\xi^\infty}\right)^2 = 1$$

$E^n \times R$ Euclidean $(n+1)$-space

$$(\vec u, v)\cdot(\vec x, y) = \vec u\cdot\vec x + vy. \qquad \vec x\cdot\vec x + y^2 = 1$$

$$\Psi^n \longrightarrow S^n \text{ (unit sphere)} \subset E^n \times R$$

$$\langle e_0\xi^0 + \vec\xi + e_\infty\xi^\infty\rangle \mapsto \left(\vec\xi\,\frac{2}{\xi^0+\xi^\infty},\ \frac{\xi^0-\xi^\infty}{\xi^0+\xi^\infty}\right)$$

Inverse of:
$$\begin{cases} \vec{x} = \vec{\mathfrak{z}}\,\dfrac{2}{\mathfrak{z}^0+\mathfrak{z}^\infty} \\[2ex] y = \dfrac{\mathfrak{z}^0-\mathfrak{z}^\infty}{\mathfrak{z}^0+\mathfrak{z}^\infty} \end{cases}$$

from
$$\begin{cases} \vec{x}\,\dfrac{\mathfrak{z}^0+\mathfrak{z}^\infty}{2} = \vec{\mathfrak{z}} \\[2ex] y\,\dfrac{\mathfrak{z}^0+\mathfrak{z}^\infty}{2} = \dfrac{\mathfrak{z}^0-\mathfrak{z}^\infty}{2} \\[2ex] 1\,\dfrac{\mathfrak{z}^0+\mathfrak{z}^\infty}{2} = \dfrac{\mathfrak{z}^0+\mathfrak{z}^\infty}{2} \end{cases} \qquad \begin{aligned} (y+1)\,\frac{\mathfrak{z}^0+\mathfrak{z}^\infty}{2} &= \mathfrak{z}^0 \\ (-y+1)\,\frac{\mathfrak{z}^0+\mathfrak{z}^\infty}{2} &= \mathfrak{z}^\infty \end{aligned}$$

gives
$$\Psi^n \longleftarrow S^n$$

$$\langle e_0(y+1) + \vec{x} + e_\infty(-y+1) \rangle \longleftarrow (\vec{x}, y)$$

Note:
$$\vec{x}\cdot\vec{x} - (y+1)(-y+1) = \vec{x}\cdot\vec{x} + y^2 - 1$$

b) Two Totally isotropic 2-planes in $\mathbb{R}^{n+3}$

$$\underbrace{\tilde{\mathfrak{z}}\cdot\tilde{\mathfrak{z}}}_{e_1\mathfrak{z}^1+\cdots+e_{n-1}\mathfrak{z}^{n-1}} + (\mathfrak{z}^n)^2 - (\mathfrak{z}^r)^2 - \mathfrak{z}^0\mathfrak{z}^\infty = 0$$

or
$$\tilde{\mathfrak{z}}\cdot\tilde{\mathfrak{z}} - \underbrace{(-\mathfrak{z}^n+\mathfrak{z}^r)(\mathfrak{z}^n+\mathfrak{z}^r)}_{\text{signatures } (-1,1)} - \underline{\mathfrak{z}^0\mathfrak{z}^\infty} = 0$$

Corresponding roles played by:

$\mathfrak{z}^r$	$\dfrac{\mathfrak{z}^0+\mathfrak{z}^\infty}{2}$
e^r	e_0+e_∞
$\langle e_r\rangle^\perp$	$\langle e_0+e_\infty\rangle^\perp$

Note: $(e_0+e_\infty|\mathfrak{z}) = -\dfrac{\mathfrak{z}^\infty+\mathfrak{z}^0}{2}$ & $(e_0+e_\infty|e_0+e_\infty) = -1$

c) A second sphere.

$$I^n = \langle e_0 + e_\infty \rangle^\perp \cap \Omega^{n+1}$$

In $\langle e_0 + e_\infty \rangle^\perp = \{ \upsilon^0 + \upsilon^\infty = 0 \}$ $\qquad \upsilon =$ upsilon

with points $\langle e_0 \upsilon^0 + \vec{\upsilon} + e_r \upsilon^r - e_\infty \upsilon^0 \rangle$.

$$\vec{\upsilon} \cdot \vec{\upsilon} - (\upsilon^r)^2 + (\upsilon^0)^2 = 0$$

If $\upsilon^r = 0$, $\vec{\upsilon} \cdot \vec{\upsilon} + (\upsilon^0)^2 = 0$ gives $\vec{\upsilon} = 0$ and $\upsilon^0 = 0$, so $\upsilon^\infty = 0$. Thus $\upsilon^r \neq 0$

Now write $\vec{\upsilon} \cdot \vec{\upsilon} - (\upsilon^r)^2 + (\upsilon^0)^2 = 0$

as
$$\frac{\vec{\upsilon}}{\upsilon^r} \cdot \frac{\vec{\upsilon}}{\upsilon^r} + \left(\frac{\upsilon^0}{\upsilon^r} \right)^2 = 1.$$

Thus

$$I^n \longrightarrow S^n \text{ (unit sphere)} \subset \mathbb{E}^n \times \mathbb{R}$$

$$\langle e_0 \upsilon^0 + \vec{\upsilon} + e_r \upsilon^r - e_\infty \upsilon^0 \rangle \rightsquigarrow \left(\frac{\vec{\upsilon}}{\upsilon^r}, \frac{\upsilon^0}{\upsilon^r} \right)$$

Inverse of $\begin{cases} \vec{u} = \dfrac{\vec{\upsilon}}{\upsilon^r} \\ v = \dfrac{\upsilon^0}{\upsilon^r} \end{cases}$ is $\begin{cases} \vec{\upsilon} = \vec{u}\, \upsilon^r \\ \upsilon^0 = v\, \upsilon^r . \end{cases}$

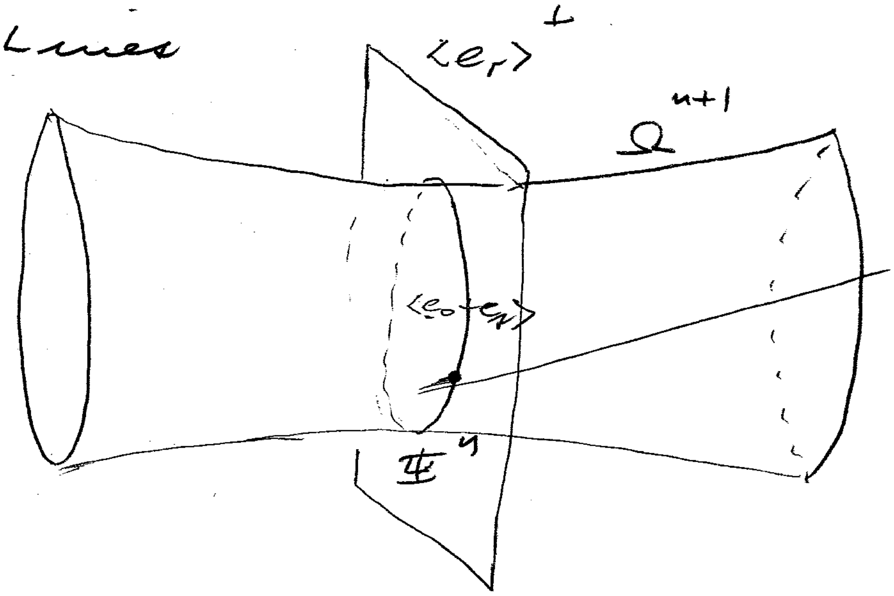
So obtain

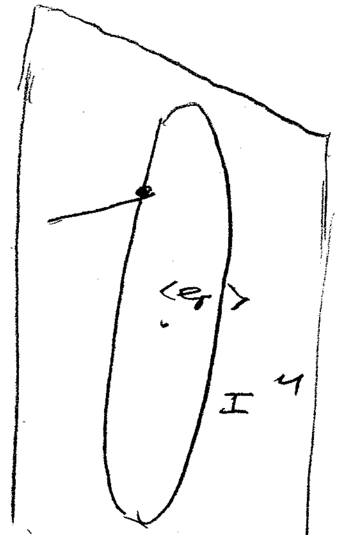
$$I^n \longleftarrow S^n$$

$$\langle e_0 v + \vec{u} + e_r - e_\infty v \rangle \longleftarrow (\vec{u}, v)$$

Note: $\vec{u} \cdot \vec{u} - 1 - (-1)v^2 = \vec{u} \cdot \vec{u} + v^2 - 1$

d) Lines





$\langle e_0+e_4\rangle^{\perp}$ at "infinity"

Note:

$$\zeta = e_0\zeta^0 + \vec{\zeta} + e_4\zeta^4 \quad \in \langle e_r\rangle^{\perp}$$

$$v = e_0 v^0 + \vec{v} + e_r v^r - e_4 v^0 \quad \in \langle e_0+e_4\rangle^{\perp}$$

Then

$$(v|\zeta) = -\tfrac{1}{2}v^0\zeta^4 + \vec{v}\cdot\vec{\zeta} + \tfrac{1}{2}v^0\zeta^0$$

$$= \tfrac{1}{2}\left(2\,\vec{v}\cdot\vec{\zeta} + v^0(\zeta^0-\zeta^4)\right)$$

while

$$\left(\frac{\vec{v}}{v^r}, \frac{v^0}{v^r}\right)\cdot\left(\vec{\zeta}\,\frac{2}{\zeta^0+\zeta^4}, \frac{\zeta^0-\zeta^4}{\zeta^0+\zeta^4}\right)$$

$$= \frac{1}{v^r(\zeta^0+\zeta^4)}\left(2\,\vec{v}\cdot\vec{\zeta} + v^0(\zeta^0-\zeta^4)\right)$$

Or:

$$\left(e_0 v + \vec{u} + e_r - e_4 v \mid e_0(y+1) + \vec{x} + e_4(-y+1)\right)$$

$$= -\tfrac{1}{2}v(-y+1) + \vec{u}\cdot\vec{x} + \tfrac{1}{2}v(y+1)$$

$$= \vec{u}\cdot\vec{x} + vy = (\vec{u}, v)\cdot(\vec{x}, y).$$

Thus: $\langle v, \mathfrak{z} \rangle \subset \Omega^{n+1}$ iff corresponding unit vectors in $E^n \times \mathbb{R}$ are orthogonal.

e) $V_2(E^n \times \mathbb{R}) = \begin{pmatrix} \text{Stiefel manifold of} \\ \text{orthonormal 2-frames} \end{pmatrix}$

$$= \left\{ \left((\vec{u}, v), (\vec{x}, y) \right) \;\middle|\; \begin{array}{l} (\vec{u}, v) \cdot (\vec{u}, v) = 1 \\ (\vec{x}, y) \cdot (\vec{x}, y) = 1 \\ (\vec{u}, v) \cdot (\vec{x}, y) = 0 \end{array} \right\}.$$

Have bijection:

$$V_2^{2n-1} \longrightarrow \Lambda^{2n-1} = \begin{pmatrix} \text{space of} \\ \text{lines in } \Omega^{n+1} \end{pmatrix}$$

$$((\vec{u}, v), (\vec{x}, y)) \mapsto \langle e_0 v + \vec{u} + e_r - e_\infty v, \; e_0(y+1) + \vec{x} + e_\infty(-y+1) \rangle.$$

f) Contact form:

$v = e_0 v + \vec{u} + e_r - e_\infty v$

$\mathfrak{z} = e_0(y+1) + \vec{x} + e_\infty(-y+1)$

$d\mathfrak{z} = e_0\, dy + d\vec{x} - e_\infty dy.$

$(v \mid d\mathfrak{z}) = \frac{1}{2} v\, dy + \vec{u}\, dx + \frac{1}{2} v\, dy$

$= \vec{u}\, d\vec{x} + v\, dy = (\vec{u}, v) \cdot d(\vec{x}, y)$

contact form on Λ^{2n-1} up to scalar

on V_2 - up to scalar.