Stereographic projection - from $\langle e_\infty \rangle$.
Make corresponding points of $\langle e_0 \rangle^\perp$ and Ω^{n+1} collinear with $\langle e_\infty \rangle$:

Quadratic map: $\langle e_0 \rangle^\perp \setminus (\langle e_0, e_\infty \rangle^\perp \cap \Omega^{n+1}) \longrightarrow \Omega^{n+1}$

$$\langle e_0 \zeta^0 + \vec{\zeta} + e_r \zeta^r \rangle \rightsquigarrow \langle e_0 (\zeta^0)^2 + \vec{\zeta}\zeta^0 + e_r \zeta^0 \zeta^r + e_\infty (\vec{\zeta}\cdot\vec{\zeta} - (\zeta^r)^2) \rangle$$

Restriction of projection

$\mathbb{P}^{n+2} \setminus \langle e_\infty \rangle \longrightarrow \langle e_0 \rangle^\perp$ by $\langle \zeta \rangle \rightsquigarrow \langle \zeta, e_\infty \rangle \cap \langle e_0 \rangle^\perp$

or

$$\langle e_0 \zeta^0 + \vec{\zeta} + e_r \zeta^r + e_\infty \zeta^\infty \rangle \rightsquigarrow \langle e_0 \zeta^0 + \vec{\zeta} + e_r \zeta^r \rangle.$$

These yield a bijective map

$$\langle e_0 \rangle^\perp \setminus \langle e_0, e_\infty \rangle^\perp \xrightarrow{\text{bij}} \Omega^{n+1} \setminus (\Omega \cap \langle e_\infty \rangle^\perp)$$

since

$$\langle e_\infty \rangle^\perp = V + \langle e_r \rangle + \langle e_\infty \rangle$$

is given in $\mathbb{P}^{n+2}$ by $\zeta^0 = 0$.

<u>Möbius quadric</u> Let

$$\Psi^n = \Omega^{n+1} \cap \langle e_r \rangle^\perp = \left\{ \langle \zeta \rangle \;\middle|\; \begin{matrix} \zeta = e_0 \zeta^0 + \vec{\zeta} + e_\infty \zeta^\infty \\ \vec{\zeta}\cdot\vec{\zeta} - \zeta^0 \zeta^\infty = 0 \end{matrix} \right\}.$$

Stereographic projection restricts to

$$(\langle e_0 \rangle^\perp \setminus \langle e_0, e_\infty \rangle^\perp) \cap \langle e_r \rangle^\perp \longrightarrow (\Omega^{n+1} \setminus (\Omega \cap \langle e_\infty \rangle^\perp)) \cap \langle e_r \rangle^\perp$$

or

$$\langle e_0, e_r \rangle^\perp \setminus \langle e_0, e_r, e_\infty \rangle^\perp \longrightarrow \Psi^n \setminus (\Psi \cap \langle e_\infty \rangle^\perp)$$

as:

$$\langle e_0 \zeta^0 + \vec{\zeta} \rangle \longrightarrow \langle e_0 (\zeta^0)^2 + \vec{\zeta}\zeta^0 + e_\infty \vec{\zeta}\cdot\vec{\zeta} \rangle$$

$\zeta^0 \neq 0$

or

$$= \langle e_0 + \frac{\vec{\zeta}}{\zeta^0} + e_\infty \frac{\vec{\zeta}}{\zeta^0}\cdot\frac{\vec{\zeta}}{\zeta^0} \rangle$$

In terms of $V \times \mathbb{R}$, Ψ^n is the paraboloid $z = \vec{x}\cdot\vec{x}$.

<u>Pole and polar</u>

Let $\langle \alpha \rangle \in \Omega^{n+1}$, so $\langle \alpha \rangle$ is an oriented sphere or Lie cycle. The prime $\langle \alpha \rangle^\perp$ is tangent to Ω^{n+1} at $\langle \alpha \rangle$, so $\Omega^{n+1} \cap \langle \alpha \rangle^\perp$ is a (projective) <u>cone</u>.

The prime $\langle \alpha \rangle^\perp = \begin{pmatrix} \text{polar of } \langle \alpha \rangle \\ \text{w.r.t. } \Omega^{n+1} \end{pmatrix}$ meets $\langle e_r \rangle^\perp$ in

$$\langle \alpha \rangle^\perp \cap \langle e_r \rangle^\perp = \langle \alpha, e_r \rangle^\perp = \begin{pmatrix} \text{polar of } \langle \alpha, e_r \rangle \cap \langle e_r \rangle^\perp \\ \text{w.r.t. } \Psi^n \end{pmatrix}.$$

For $\alpha = e_0\alpha^0 + \vec{\alpha} + e_r\alpha^r + e_\ast\alpha^\ast$ in Ω^{n+1} we have:

$$\langle\alpha\rangle^\perp: \quad \vec{\alpha}\cdot\vec{\zeta} - \alpha^r\zeta^r - \tfrac{1}{2}(\alpha^0\zeta^\ast + \alpha^\ast\zeta^0) = 0$$

$$\langle e_r\rangle^\perp: \quad \zeta^r = 0$$

so

$$\langle\alpha, e_r\rangle^\perp \text{ is } \quad \vec{\alpha}\cdot\vec{\zeta} - \tfrac{1}{2}(\alpha^0\zeta^\ast + \alpha^\ast\zeta^0) = 0$$

or

$$(e_0\alpha^0 + \vec{\alpha} + e_\ast\alpha^\ast \mid \zeta) = 0 \qquad \langle\zeta\rangle \text{ in } \mathbb{P}^{n+2} \text{ or in } \langle e_r\rangle^\perp.$$

Since

$$(e_0\alpha^0 + \vec{\alpha} + e_\ast\alpha^\ast \mid \text{— " —})$$
$$= \lambda_0(\vec{\alpha}\cdot\vec{\alpha} - \alpha^0\alpha^\ast) = \lambda_0(\alpha^r)^2,$$

the point

$$\langle\alpha, e_r\rangle \cap \langle e_r\rangle^\perp = \langle e_0\alpha^0 + \vec{\alpha} + e_\ast\alpha^\ast\rangle \text{ in } \langle e_r\rangle^\perp$$

is an element of

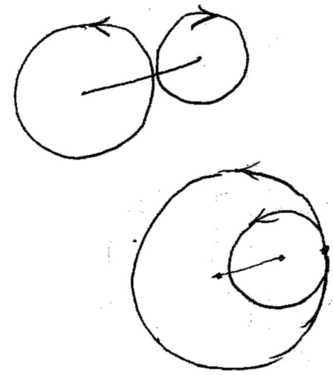
$$\Psi^+ = \{\langle\zeta\rangle \in \langle e_r\rangle^\perp \mid (\zeta|\zeta)/\lambda_0 \in (h^\ast)^2\}.$$

<u>Note</u>: α^r and $-\alpha^r$ give same point of $\langle e_r\rangle^\perp$, so

$$\Omega^{n+1} \setminus \langle e_r\rangle^\perp \longrightarrow \Psi^+$$
$$\langle\alpha\rangle \longmapsto (\langle\alpha\rangle + \langle e_r\rangle)^\perp \cap \langle e_r\rangle^\perp$$

is 2-to-1.

<u>Tangency</u>



The two hyperspheres

$(\vec{x}-\vec{a})\cdot(\vec{x}-\vec{a}) = (a^r)^2$

$(\vec{x}-\vec{b})\cdot(\vec{x}-\vec{b}) = (b^r)^2$

of V are <u>tangent</u> if

$$(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = (a^r \pm b^r)^2$$

Take the radius into account and use $a^r - b^r$. Complete squares to obtain

$$\underbrace{\vec{a}\cdot\vec{a} - (a^r)^2}_{a^\infty} + \underbrace{\vec{b}\cdot\vec{b} - (b^r)^2}_{b^\infty} = 2\,\vec{a}\cdot\vec{b} - 2a^r b^r$$

or $\vec{a}\cdot\vec{b} - a^r b^r - \frac{1}{2}a^\infty - \frac{1}{2}b^\infty = 0$

Made homogeneous using $\vec{a} = \frac{\vec{\alpha}}{\alpha^0}$, etc., this is

$$\vec{\alpha}\cdot\vec{\beta} - \alpha^r\beta^r - \frac{1}{2}(\alpha^0\beta^\infty + \alpha^\infty\beta^0) = 0$$

<u>Hence</u>: Lie cycles $\langle\alpha\rangle$ and $\langle\beta\rangle$ in Ω^{n+1} are in <u>oriented contact</u> if $(\alpha|\beta) = 0$, or equivalently the line $\langle\alpha,\beta\rangle$ lies in Ω^{n+1}.

<u>Note</u>: $\langle\alpha\rangle + \langle\beta\rangle = \langle\alpha,\beta\rangle$ is a <u>parabolic pencil</u> of cycles in oriented contact.

Cones

Notation: $\quad \xi = e_0 \xi^0 + \vec{\xi} + e_r \xi^r + e_* \xi^*$

Set $\quad \hat{\xi} = e_0 \xi^0 + \vec{\xi} \qquad + e_* \xi^*$

Note that for $\langle \xi \rangle \neq \langle e_r \rangle$:

$$(\langle \xi \rangle + \langle e_r \rangle) \cap \langle e_r \rangle^\perp = \langle \hat{\xi} \rangle .$$

The projection is

$$\begin{aligned} \Omega^{n+1} &\longrightarrow \Psi^n \cup \Psi^+ \\ \langle \alpha \rangle &\longmapsto \langle \hat{\alpha} \rangle \end{aligned}$$

Observe:

$$(\xi \mid \eta) = (\hat{\xi} \mid \hat{\eta}) - \lambda_0 \xi^r \eta^r .$$

Now: Suppose $\langle \alpha, \xi \rangle \subset \Omega^{n+1}$.

Then

$$\begin{aligned} (\alpha \mid \alpha) &= 0 \quad &\text{so} \quad (\hat{\alpha} \mid \hat{\alpha}) &= \lambda_0 (\alpha^r)^2 \\ (\alpha \mid \xi) &= 0 \quad &\text{so} \quad (\hat{\alpha} \mid \hat{\xi}) &= \lambda_0 \alpha^r \xi^r \\ (\xi \mid \xi) &= 0 \quad &\text{so} \quad (\hat{\xi} \mid \hat{\xi}) &= \lambda_0 (\xi^r)^2 \end{aligned}$$

and hence

$$(\hat{\alpha} \mid \hat{\xi})^2 - (\hat{\alpha} \mid \hat{\alpha})(\hat{\xi} \mid \hat{\xi}) = 0 .$$

That is, the line $\langle \hat{\alpha}, \hat{\xi} \rangle$ in $\langle e_r \rangle^\perp$ is tangent to Ψ^n.

Consequence. The projection from $\langle e_r \rangle$ sends the cone $\Omega^{n+1} \cap \langle \alpha \rangle^\perp$ with vertex $\langle \alpha \rangle$ in Ω^{n+1} to the cone of $\langle e_r \rangle^\perp$ tangent to $\Psi = \Omega \cap \langle e_r \rangle^\perp$ and having vertex $\langle \hat{\alpha} \rangle$ in $\Psi \cup \Psi^+$.