<u>Relative powers of oriented spheres</u>

1) The <u>relative Laguerre power</u> of two oriented hyperspheres of V:

$$(\vec{x}-\vec{a})\cdot(\vec{x}-\vec{a}) = (a^r)^2$$

and

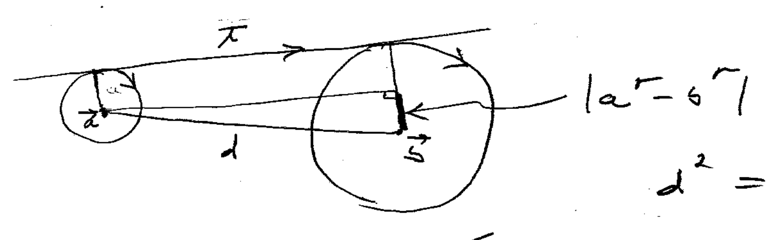
$$(\vec{x}-\vec{b})\cdot(\vec{x}-\vec{b}) = (b^r)^2$$

is

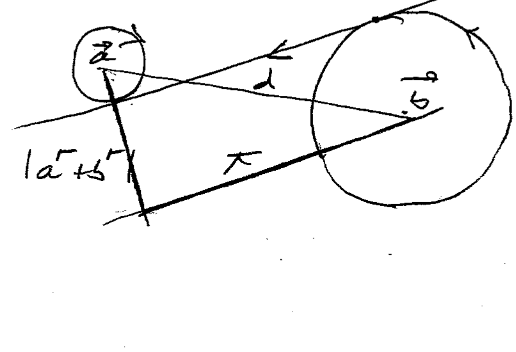
$$\Upsilon = (\vec{a}-\vec{b})^2 - (a^r-b^r)^2.$$

Note: 1) This depends on the relative signs of the radii a^r and b^r.
2) If $b^r = 0$, this reduces to the classical Steiner power of the point $\vec{b}$ with respect to the first hypersphere.

<u>Remark</u> For $V = \mathbb{R}^n(\text{col})$ Euclidean space, Υ is <u>square</u> of the <u>tangential distance</u> between the oriented hyperspheres.



$$d^2 = \tau^2 + (a^r-b^r)^2$$



$$d^2 = \tau^2 + (a^r+b^r)^2$$

Rewrite relative Laguerre power as

$$\Upsilon = \vec{a}\cdot\vec{a} - (a^-)^2 + \vec{b}\cdot\vec{b} - (b^-)^2$$
$$- 2\vec{a}\cdot\vec{b} + 2a^-b^-$$
$$= a^+ + b^+ - 2\vec{a}\cdot\vec{b} + 2a^-b^-$$
$$= -2\left(\vec{a}\cdot\vec{b} - a^-b^- - \tfrac{1}{2}a^+ - \tfrac{1}{2}b^+\right)$$

Make homogeneous by $\vec{a} = \vec{\alpha}/\alpha^0$ etc.!

$$\Upsilon = -2\left(\frac{\vec{\alpha}}{\alpha^0}\cdot\frac{\vec{\beta}}{\beta^0} - \frac{\alpha^-}{\alpha^0}\frac{\beta^-}{\beta^0} - \frac{1}{2}\frac{\alpha^+}{\alpha^0} - \frac{1}{2}\frac{\beta^+}{\beta^0}\right)$$

Setting $\alpha = e_0\alpha^0 + \vec{\alpha} + e_r\alpha^- + e_+\alpha^+$ and likewise β, we have:

$$\Upsilon = -2\,\frac{(\alpha|\beta)/\lambda_0}{\alpha^0\beta^0}$$

Note

$$(e_+|\alpha) = -\frac{\lambda_0}{2}\alpha^0 \quad\text{and}\quad (e_+|\beta) = -\frac{\lambda_0}{2}\beta^0$$

so

$$\Upsilon = -\frac{1}{2}\,\frac{\lambda_0(\alpha|\beta)}{(e_+|\alpha)(e_+|\beta)}.$$

Note! e_+ is a fixed choice in W; there is no need to consider "rescaling"!

Notation: $\Upsilon(\langle\alpha\rangle, \langle\beta\rangle)$.

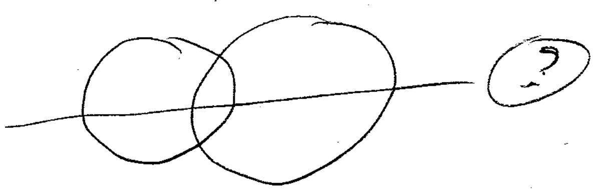
Cross ratio

$$\Upsilon(\langle\alpha\rangle, \langle\beta\rangle) = -\frac{1}{2}\,\frac{\lambda_0(\alpha|\beta)}{(e_2|\alpha)(e_2|\beta)}$$

$$\frac{\Upsilon(\langle\alpha\rangle, \langle\zeta\rangle)}{\Upsilon(\langle\zeta\rangle, \langle\beta\rangle)} = -\frac{1}{2}\,\frac{\lambda_0(\alpha|\zeta)}{(e_2|\alpha)(e_2|\zeta)} \Big/ -\frac{1}{2}\,\frac{\lambda_0(\beta|\zeta)}{(e_2|\beta)(e_2|\zeta)}$$

$$= \frac{(\alpha|\zeta)}{(e_2|\alpha)} \Big/ \frac{(\beta|\zeta)}{(e_2|\beta)}$$ - eliminates $\langle e_2\rangle$ rescaling

$$\frac{\Upsilon(\langle\alpha\rangle, \langle\zeta\rangle)}{\Upsilon(\langle\zeta\rangle, \langle\beta\rangle)} \Big/ \frac{\Upsilon(\langle\alpha\rangle, \langle\eta\rangle)}{\Upsilon(\langle\eta\rangle, \langle\beta\rangle)} = \frac{(\alpha|\zeta)}{(\zeta|\beta)} \Big/ \frac{(\alpha|\eta)}{(\eta|\beta)}$$



$$\Upsilon(\langle\alpha\rangle, \langle\beta\rangle) = -\frac{1}{2}\,\frac{(\alpha|\beta)}{(e_2|\alpha)(e_2|\beta)}$$

$$\Upsilon(\langle\lambda\rangle, \langle\mu\rangle) = \underline{\qquad}$$

$$\frac{\Upsilon(\langle\alpha\rangle, \langle\beta\rangle)}{\Upsilon(\langle\lambda\rangle, \langle\mu\rangle)} = \frac{(\alpha|\beta)}{(e_2|\alpha)(e_2|\beta)}\,\frac{(e_2|\lambda)(e_2|\mu)}{(\lambda|\mu)}$$

<u>Bunches of Lie cycles</u>

Fix a Lie cycle $\langle\alpha\rangle \in \Omega^{n+1} - \langle e^*\rangle^\perp$ (so $\alpha^0 \neq 0$) and a scalar $a \in k$. The <u>bunch</u> of Lie cycles with <u>central cycle</u> $\langle\alpha\rangle$ and <u>power</u> a is

$$\{ \langle\mathfrak{z}\rangle \in \Omega^{n+1} \setminus \langle e_*\rangle^\perp \mid \Upsilon(\langle\alpha\rangle, \langle\mathfrak{z}\rangle) = a \}.$$

The condition $\Upsilon(\langle\alpha\rangle, \langle\mathfrak{z}\rangle) = a$ is

$$-\frac{1}{2}\,\frac{\lambda_0(\alpha|\mathfrak{z})}{(e_*|\alpha)(e_*|\mathfrak{z})} = a,$$

which we may rewrite

$$\left(\alpha + e_* 2a\frac{(e_*|\alpha)}{\lambda_0} \,\Big|\, \mathfrak{z}\right) = 0.$$

Thus, the bunch with central cycle $\langle\alpha\rangle$ and power a is

$$\Omega^{n+1} \cap \underbrace{\left\langle \alpha + e_* 2a\frac{(e_*|\alpha)}{\lambda_0}\right\rangle^\perp}_{\dim = n+1 \text{ in } \mathbb{P}^{n+2}}$$

Note

$$\left(\alpha + e_* 2a\frac{(e_*|\alpha)}{\lambda_0} \,\Big|\, \text{— ditto —}\right)$$

$$= \underbrace{(\alpha|\alpha)}_{0} + 2(e_*|\alpha)\,2a\frac{(e_*|\alpha)}{\lambda_0} + \underbrace{(e_*|e_*)}_{0}(\cdots)$$

$$= 4a\frac{(e_*|\alpha)^2}{\lambda_0}$$

is $\neq 0$ or $=0$ as a is $\neq 0$ or $=0$, by $\langle\alpha\rangle \notin \langle e_*\rangle^\perp$.

And!
$$\left(e_* \,\Big|\, \alpha + e_* 2a\frac{(e_*|\alpha)}{\lambda_0}\right) = (e_*|\alpha) \neq 0.$$

i.e. $$\left\langle \alpha + e_* 2a\frac{(e_*|\alpha)}{\lambda_0}\right\rangle \in \mathbb{P}^{n+2} \setminus \langle e_*\rangle^\perp.$$

Consequently, obtain the map

$$\begin{pmatrix}\text{bunches}\\ \text{of Lie}\\ \text{cycles}\end{pmatrix} \xrightarrow{\text{bijection}} \mathbb{P}^{n+2} \setminus \langle e_\ast\rangle^\perp$$

$$\begin{pmatrix}\text{bunch with}\\ \text{central cycle } \langle\alpha\rangle \in \Omega \setminus \langle e_\ast\rangle^\perp\\ \text{and power } a \in h\end{pmatrix} \rightsquigarrow \left\langle \alpha + e_\ast 2a \frac{(e_\ast|\alpha)}{\lambda_0} \right\rangle.$$

The inverse is obtained as follows:
Let $\langle\beta\rangle \in \mathbb{P}^{n+2} \setminus \langle e_\ast\rangle^\perp$. Then the line $\langle e_\ast\rangle + \langle\beta\rangle = \langle e_\ast, \beta\rangle$ does not lie in Ω by $(e_\ast|\beta) \neq 0$. Thus $\langle e_\ast, \beta\rangle$ meets Ω in two points, $\langle e_\ast\rangle$ and $\langle\alpha\rangle$ where $\alpha = \beta - e_\ast x$, $x \in h$ ($x \neq \infty$ to exclude $\langle e_\ast\rangle$ a second time).
Then $(e_\ast|\alpha) = (e_\ast|\beta)$ is $\neq 0$ so $\langle\alpha\rangle \in \Omega \setminus \langle e_\ast\rangle^\perp$,
And

$$\beta = \alpha + e_\ast 2\left(\frac{x\lambda_0}{2(e_\ast|\alpha)}\right)\frac{(e_\ast|\alpha)}{\lambda_0}$$

shows the power for the bunch is $\dfrac{x\lambda_0}{2(e_\ast|\alpha)}$.