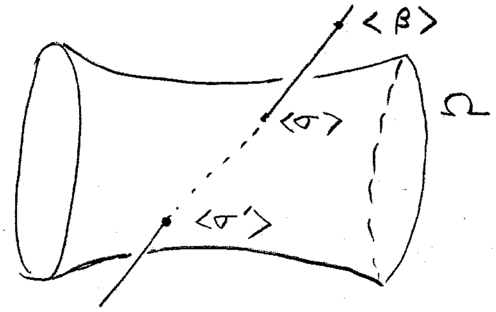
<u>Lie contact inversion</u>




Let $\Omega$ be <u>any</u> quadric in a projective space and let $\langle\beta\rangle \notin \Omega$ be a point <u>off</u> $\Omega$.

Let $\langle\sigma\rangle \in \Omega$ be arbitrary.

Then, the following are <u>equivalent</u>:

1) $\sigma' = \sigma - \beta \dfrac{2(\beta|\sigma)}{(\beta|\beta)}$

2) $\langle\sigma'\rangle$ is the harmonic conjugate of $\langle\sigma\rangle$ with respect to $\langle\beta\rangle$ and $\langle\beta,\sigma\rangle \cap \langle\beta\rangle^\perp$ on the line $\langle\beta\rangle + \langle\sigma\rangle = \langle\beta,\sigma\rangle$

3) $\langle\sigma'\rangle \in \Omega$ is the second of the two points of $\Omega$ (possibly equal) collinear with $\langle\beta\rangle$ and $\langle\sigma\rangle$: $\langle\beta\rangle + \langle\sigma\rangle = \langle\beta\rangle + \langle\sigma'\rangle$.

4) $\langle\sigma'\rangle \in \Omega$ is the second of the two points of $\Omega$ for which $\langle\beta\rangle^\perp \cap \langle\sigma\rangle^\perp = \langle\beta\rangle^\perp \cap \langle\sigma'\rangle^\perp$.
(Dual of 3.)

And these imply:

5) $\langle\sigma'\rangle \in \Omega$ is the second of the two points of $\Omega$ for which

$$\{\langle\mathfrak{z}\rangle \in \Omega \mid (\beta|\mathfrak{z}) = 0 \text{ and } (\sigma|\mathfrak{z}) = 0\}$$
$$= \{\langle\mathfrak{z}\rangle \in \Omega \mid (\beta|\mathfrak{z}) = 0 \text{ and } (\sigma'|\mathfrak{z}) = 0\}.$$

(Intersection of $\Omega$ and 4.)

In case that $\Omega$ is the Lie quadric and $\langle\beta\rangle = \langle \alpha + e_* 2a \frac{(e_*|\alpha)}{\lambda_0} \rangle \in \mathbb{P}^{n+2} \setminus \langle e_* \rangle^\perp$ is the bunch with central cycle $\langle\alpha\rangle \in \Omega \setminus \langle e_*\rangle^\perp$ and power $a \in k$, in addition:

6) $\{ \langle\zeta\rangle \in \Omega \mid \Upsilon(\langle\alpha\rangle, \langle\zeta\rangle) = a$ and $\Upsilon(\langle\sigma\rangle, \langle\zeta\rangle) = 0 \}$

$= \{ \langle\zeta\rangle \in \Omega \mid$ ——— and $\Upsilon(\langle\sigma'\rangle, \langle\zeta\rangle) = 0 \}$.

(By Formula on p. 15.)

<u>Def</u> (Yaglom). Given a bunch $\langle\beta\rangle$ of Lie cycles, the <u>contact inversion</u> with respect to the bunch $\langle\beta\rangle$ is the transformation sending a Lie cycle $\langle\sigma\rangle$ to the second of the two Lie cycles $\langle\sigma'\rangle$ such cycles of $\langle\beta\rangle$ which are tangent to $\langle\sigma\rangle$ are also tangent to $\langle\sigma'\rangle$ (Possibly $\langle\sigma'\rangle$ coincides with $\langle\sigma\rangle$.).

<u>N.B.</u> Here we will include all $\langle\beta\rangle \in \mathbb{P}^{n+2} - \Omega^{n+1}$ among contact inversions by writing

$$\sigma' = \sigma - \beta \frac{2(\beta|\sigma)}{(\beta|\beta)}$$

<u>Remark</u>. One can check by computation that $\Upsilon(\langle\alpha\rangle, \langle\sigma\rangle) \cdot \Upsilon(\langle\alpha\rangle, \langle\sigma'\rangle) = a^2$, when $\beta = \alpha + e_* 2a \frac{(e_*|\alpha)}{\lambda_0}$.

<u>Appendix</u> Verification of an identity.

$$\Upsilon(\langle\alpha\rangle,\langle\sigma\rangle) = -\frac{1}{2}\,\frac{\lambda_0(\alpha|\sigma)}{(e_+|\alpha)(e_+|\sigma)}\,, \qquad \langle\alpha\rangle,\langle\sigma\rangle \in \Omega$$

$$\sigma' = \sigma - \left(\alpha + e_+ 2a\,\frac{(e_+|\alpha)}{\lambda_0}\right)\frac{2\left(\alpha + e_+ 2a\,\frac{(e_+|\alpha)}{\lambda_0}\,\middle|\,\sigma\right)}{\left(\alpha + e_+ 2a\,\frac{(e_+|\alpha)}{\lambda_0}\,\middle|\,-\,\text{''}\,-\right)}$$

$$= \sigma - \left(\alpha + e_+ 2a\,\frac{(e_+|\alpha)}{\lambda_0}\right)\frac{2\left((\alpha|\sigma) + 2a\,\frac{(e_+|\sigma)(e_+|\alpha)}{\lambda_0}\right)}{4a\,\frac{(e_+|\alpha)^2}{\lambda_0}}$$

so

$$(\alpha|\sigma') = \cancel{(\alpha|\sigma)} - \left(2a\,\frac{(e_+|\alpha)^2}{\lambda_0}\right)\frac{2\left(\cancel{(\alpha|\sigma)} + 2a\,\frac{(e_+|\sigma)(e_+|\alpha)}{\lambda_0}\right)}{4a\,\frac{(e_+|\alpha)^2}{\lambda_0}}$$

$$= -2a\,\frac{(e_+|\sigma)(e_+|\alpha)}{\lambda_0}$$

and

$$(e_+|\sigma') = \cancel{(e_+|\sigma)} - (e_+|\alpha)\,\frac{2\left((\alpha|\sigma) + \cancel{2a\,\frac{(e_+|\sigma)(e_+|\alpha)}{\lambda_0}}\right)}{4a\,\frac{(e_+|\alpha)^2}{\lambda_0}}$$

$$= -\frac{\lambda_0}{2a}\,\frac{(\alpha|\sigma)}{(e_+|\alpha)}$$

Thus

$$-\frac{1}{2}\,\frac{\lambda_0(\alpha|\sigma)}{(e_+|\alpha)(e_+|\sigma)}\cdot -\frac{1}{2}\,\frac{\lambda_0(\alpha|\sigma')}{(e_+|\alpha)(e_+|\sigma')}$$

$$= \frac{\cancel{\lambda_0^2}}{4}\,\frac{\cancel{(\alpha|\sigma)}}{\cancel{(e_+|\alpha)}\cancel{(e_+|\sigma)}}\;\frac{-2a\,\frac{\cancel{(e_+|\sigma)}\cancel{(e_+|\alpha)}}{\cancel{\lambda_0}}}{\cancel{(e_+|\alpha)}\cdot -\frac{\cancel{\lambda_0}}{2a}\,\frac{\cancel{(\alpha|\sigma)}}{\cancel{(e_+|\alpha)}}} = a^2$$

Ok!

$$\Upsilon(\langle\alpha\rangle,\langle\sigma\rangle)\,\Upsilon(\langle\alpha\rangle,\langle\sigma'\rangle) = a^2.$$

Remark. Let $\langle \alpha_1 \rangle, \dots, \langle \alpha_{n+1} \rangle \in \Omega^{n+1}$ be "independent". Then

$$\{ \langle \xi \rangle \in \Omega \mid \Upsilon(\langle \alpha_1 \rangle, \langle \xi \rangle) = \dots = \Upsilon(\langle \alpha_{n+1} \rangle, \langle \xi \rangle) \}$$

is a row of cycles. It is a 1-parameter family of cycles. If $t$ is the common relative power, then the row is

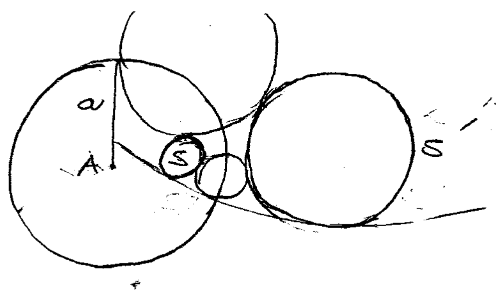
$$\Omega^{n+1} \cap \underbrace{\left\langle \alpha_1 + e_s 2t \frac{(e_s|\alpha)}{\lambda_0}, \dots, \alpha_{n+1} + e_s 2t \frac{(e_s|\alpha)}{\lambda_0} \right\rangle^{\perp}}_{\text{line of } \mathbb{P}^{n+2}}$$

show $\Upsilon(\sigma)\Upsilon(\sigma') = a^2$
determine $\sigma'$.

$\boxed{?}$

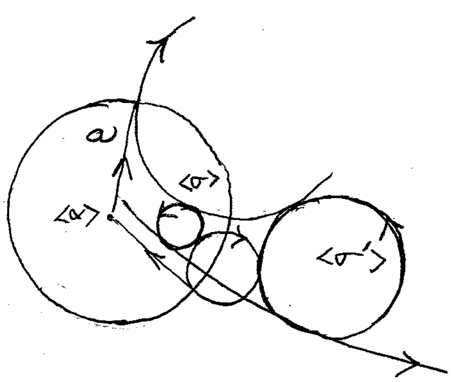
Some pictures for $n=2$.

1) Classical Möbius inversion.



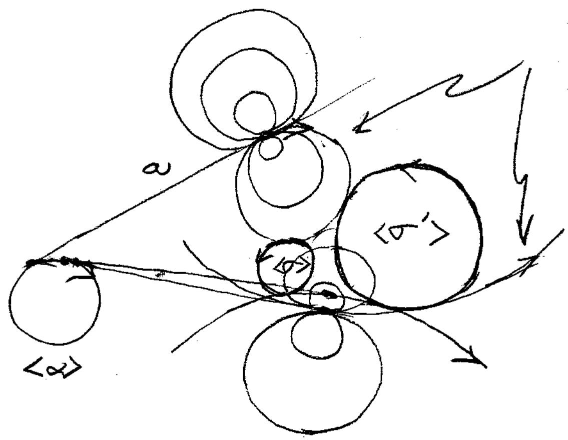

Möbius inversion in the circle with center $A$ and radius $a$ may be described by: $S$ is sent to the unique circle $S'$ such that any circle tangent to $S$ and a radius of the given circle (so is orthogonal to the given circle) is also tangent to $S'$.

2) Möbius inversion of cycles




Place orientations on the circles to obtain cycles.

## 3) Lie contact inversion of cycles




Cycles of the bunch specified by cycle ⟨α⟩ and power $a \neq 0$.

Replace point circle by cycle ⟨α⟩.

Or: All cycles of the bunch

$\{ \langle \zeta \rangle \mid T(\langle \alpha \rangle, \langle \zeta \rangle) = a \} = \Omega \cap \langle \alpha + e_* 2a \frac{(e_*|\alpha)}{\lambda_0} \rangle^{\perp}$

which have oriented contact with ⟨σ⟩ have ⟨σ⟩ as an "envelope" and ⟨σ'⟩ as a second "envelope",

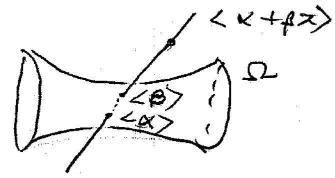
i.e. $\Omega \cap \langle \beta \rangle^{\perp} \cap \langle \sigma \rangle^{\perp} = \Omega \cap \langle \beta \rangle^{\perp} \cap \langle \sigma' \rangle$.

– cf. p. 18 4) and 5).

<u>Lie contact inversions</u> - continued.

Let $\langle\alpha\rangle, \langle\beta\rangle$ be two Lie cycles which are <u>not</u> tangent i.e. $(\alpha|\beta)\neq 0$ or $\langle\alpha,\beta\rangle \not\subset \Omega$. For $\tau \in \mathbb{R}$:

$$\langle \alpha + \beta\tau \rangle = \begin{cases} \langle\alpha\rangle & \tau = 0 \\ \langle\beta\rangle & \tau = \infty \\ \notin \Omega & \tau \neq 0,\infty \end{cases}$$




so

$$\langle \xi \rangle \rightsquigarrow \left\langle \xi - (\alpha+\beta\tau)\frac{2(\alpha+\beta\tau|\xi)}{(\alpha+\beta\tau|\alpha+\beta\tau)} \right\rangle$$

is a <u>pencil of contact inversions</u> ($\tau \neq 0,\infty$) each inversion of which

i) <u>interchanges</u> $\langle\alpha\rangle$ and $\langle\beta\rangle$
- easy to check using $(\alpha+\beta\tau|\alpha+\beta\tau) = 2(\alpha|\beta\tau)\neq 0$

ii) <u>fixes</u> any cycle in oriented contact with both $\langle\alpha\rangle$ and $\langle\beta\rangle$.
- <u>i.e.</u> fixes points of $\Omega \cap \langle\alpha\rangle^\perp \cap \langle\beta\rangle^\perp$

Pictures in $\mathbb{R}^2$

conic of Lie geom

<u>Eg</u> $n=2$ $\Omega^3 \subset \mathbb{P}^4$

$\dim(\langle\alpha,\beta\rangle^\perp) = 2$

$\Omega^3 \cap \langle\alpha,\beta\rangle^\perp$ is a <u>conic</u> in the plane $\langle\alpha,\beta\rangle^\perp$.

<u>Ref</u>

Benz p. 256

Blaschke p. 210