An algorithm for determining circles and spheres satisfying conditions which generalize tangency

Jay P. Fillmore
Department of Mathematics
University of California
at San Diego
LaJolla, CA 92093

Arthur Springer
Department of Mathematics
San Diego State University
San Diego, CA 92182

Running head: Conditions which generalize tangency

Abstract By representing each oriented circle as a point of a certain quadric in a four-dimensional projective space, it is possible to greatly generalize the Classical Apollonius Problem of determining circles that "touch" three given circles. The condition that a circle touch a given circle corresponds to a point lying additionally on a three-dimensional projective subspace. The intersection of three such subspaces is a line which

intersects the quadric in at most two points. These points represent the oriented circles which are the solutions. Many other conditions on an oriented circle can be expressed by a point lying in a three-dimensional projective subspace. These include: having a given tangential distance with respect to a given oriented circle, having a given angle with a given oriented circle, having center on a given line, and having a given signed radius. This method generalizes immediately to determining oriented spheres in space.

0. Introduction

The classical Apollonius contact problem is to determine the circles tangent to three given circles. The number of solutions is at most eight and depends on the sizes and relative positions of the circles.

This problem has been considered frequently throughout the history of geometry, both from the viewpoint of construction using ruler and compass and from that of calculation using algebra. In recent years computer graphics work, such as that of Langlet [4] and Rohne [7], has made use of linear algebra. Rigby [6] (implicitly) and Sevici [9] extend this approach using the "higher sphere geometry" of Sophus Lie [3]. It is the purpose of this paper to show how to use this geometry of oriented spheres to obtain systematic generalizations both of the conditions imposed and of the dimension of the space. These generalizations give unexpected simple methods to calculate positions of lines and circles and of planes and spheres in a variety of useful configurations.

A circle may be oriented by choosing its outward or inward unit normal field. Oriented circles which are tangent and whose unit normals at the point of tangency agree will be said to <u>touch</u>. This notion of touch extends immediately to oriented line and oriented circle. Two oriented lines touch if they are parallel and have normals that point in the same direction. A point touches an oriented circle or line if it lies on it. Let us write "oriented circle" to mean: "oriented circle, oriented line, or point". The problem of determining all "oriented circles" which touch three given "oriented circles" is by far simpler than the classical Apollonius problem. This formulation admits a separation of the problem into a part using only linear algebra and a part which requires the solving of a single quadratic equation.

Using linear algebra permits the inclusion of geometric conditions which generalize "touch a given oriented circle" - and herein lies the power of the approach.

Finally, the problem of unoriented circles can be handled by considering all possible orientations of unoriented circles.

In this paper, after discussing the underlying ideas, we formally state the algorithm together with a table of several geometric conditions, then give examples in two dimensions, and lastly give examples of the generalization to three dimensions.

The algorithm presented here was used by the authors to obtain the very precise figures which appear in their papers [1] and [2].

1. Some geometry. The mathematics behind this algorithm is a "representational geometry" created by Sophus Lie: oriented circles correspond to points on a three-dimensional quadric in four-dimensional projective space. Here we sketch just enough of the key ideas to make the genesis of the algorithm evident. We will place the emphasis on setting up calculations efficiently in terms of linear algebra. The use of "homogeneous coördinates" is central to the development of algebraic projective geometry, but here may be viewed just as an extra degree of freedom that allows one to make normalizations to simplify calculations and make them more accurate. The interested reader may consult [5] or [8] for projective geometry, [2] or [6] for the Lie geometry of circles, and Klein [3] for sphere geometry from the perspective of someone who worked at the time of Lie.

<u>1.1</u>. A <u>point</u> in four-dimensional projective space $\mathbf{P}^4$ is a one-dimensional subspace of the vector space of five-tuples of real numbers. (One could visualize a line through the origin of five dimensional space.) A point of $\mathbf{P}^4$ is described by giving a non-zero vector in the one-dimensional subspace. We will write its five components as a column vector or 5 × 1 matrix ${}^t[\ p_0,\ p_1,\ p_2,\ p_r,\ p_s\ ]$ (superscript t denotes transpose) for the purposes of matrix calculations later. Now, the same one-dimensional subspace is equally well described by giving the vector

$$ {}^t[\ p_0,\ p_1,\ p_2,\ p_r,\ p_s\ ]c = {}^t[\ p_0c,\ p_1c,\ p_2c,\ p_rc,\ p_sc\ ] $$

where c is a real number not zero. Only the ratios of the components of the vector are significant. These components p_i are called <u>homogeneous coördinates</u> of the point of $\mathbf{P}^4$.

Every condition on points describing oriented spheres, and the other various geometric objects which we consider, will be given by a linear equation like:

$$ \ell_0p_0 + \ell_1p_1 + \ell_2p_2 + \ell_rp_r + \ell_sp_s = 0 \ . $$

Again, this is unaffected by multiplying all p_i by the "scale factor" c, or all ℓ_i by a scale factor. Due to the scaling, the number of "degrees of freedom" remaining to describe points is four. This gives $\mathbf{P}^4$ its dimension.

The points of $\mathbf{P}^4$ whose homogeneous coördinates p_i satisfy a single linear equation lie in a subspace of dimension one less. This projective version of a "hyperplane" will be called a prime.

It is this scale factor c that can be chosen to normalize the components in any convenient manner. For example, one could ask that the sum of the squares of the five components is one.

Remark. In many graphics problems there is no loss of sharpness when all coördinates are written as fractions with reasonably small denominators. Normalizing these coördinates to be integers then insures that all calculations will be done with simple exact arithmetic. This we will do in the examples later.

1.2 In the Euclidean plane with ordinary point coördinates (x_1, x_2), the equation of a line with unit normal vector (n_1, n_2) and signed distance from the origin n_s is

$$n_1x_1 + n_2x_2 = n_s .$$

The normal (n_1, n_2) describes the orientation of this line; the signed distance n_s is measured from the origin to the line, with positive being in the direction of the unit normal. The three parameters n_1, n_2, n_s determine the point:

$$A = {}^t[0, n_1, n_2, -1, n_s]$$

of projective space $\mathbf{P}^4$. (The line passing through (2,0) and having its unit normal in the direction of the positive x_1-axis is given by ${}^t[0, 1, 0, -1, 2]$.) This describes a particular orientation of the line. The other orientation is obtained by replacing -1 by $+1$. We will say that this point A of $\mathbf{P}^4$ represents this oriented line in the Euclidean plane. In terms of homogeneous coördinates the same point of $\mathbf{P}^4$ is given as

$${}^t[0, p_1, p_2, p_r, p_s]$$

where $p_1^2 + p_2^2 - p_r^2 = 0$. Here we have multiplied by the non-zero scale factor $c = -p_r$. One obtains the original inhomogeneous coördinates from
$n_1 = p_1/(-p_r)$, $n_2 = p_2/(-p_r)$, and $n_s = p_s/(-p_r)$.

1.3 In the Euclidean plane with the usual point coördinates (x_1, x_2) , the equation of a circle with center (a_1, a_2) , and the signed radius a_r is

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 = a_r^2 .$$

Our convention will be: a circle of radius $|a_r|$ will have outward normal if $a_r > 0$ and inward normal if $a_r < 0$. This is consistent with the orientation of lines defined previously. Now, the equation of the circle may also be expanded as

$$x_1^2 + x_2^2 - 2a_1x_1 - 2a_2x_2 + 2a_s = 0$$

where $2a_s$ is the Steiner power (of the origin). These parameters are related by

$$a_1^2 + a_2^2 - a_r^2 - 2a_s = 0 .$$

The four parameters a_1, a_2, a_r, a_s determine the point

$$A = {}^t[\ 1,\ a_1,\ a_2,\ a_r,\ a_s\ ]$$

of projective space $\mathbf{P}^4$. This describes a particular orientation of the circle. The other orientation is obtained by changing the sign of a_r . We will say that this point A of $\mathbf{P}^4$ <u>represents</u> this oriented circle in the Euclidean plane. In terms of homogeneous coördinates the same point of $\mathbf{P}^4$ is given as

$${}^t[\ p_0,\ p_1,\ p_2,\ p_r,\ p_s\ ]\ ,$$

where $p_1^2 + p_2^2 - p_r^2 - 2p_0p_s = 0$. This latter condition insures that the ratios p_1/p_0, p_2/p_0 give the center of the circle, the ratio p_r/p_0 gives its signed radius (and thus its orientation), and $2p_s/p_0$ gives the Steiner power.

<u>1.4</u> The point ${}^t[\ 1,\ a_1,\ a_2,\ 0,\ a_s\ ]$ of $\mathbf{P}^4$, where $a_1^2 + a_2^2 - 2a_s = 0$, represents a point of the Euclidean plane. It is viewed as a circle of radius zero - and carrying no orientation.

1.5 With the conventions above, two oriented circles represented by

$$A = {}^t[\ 1,\ a_1,\ a_2,\ a_r,\ a_s\ ]$$

and

$$B = {}^t[\ 1,\ b_1,\ b_2,\ b_r,\ b_s\ ]$$

touch when

$$(b_1 - a_1)^2 + (b_2 - a_2)^2 = (b_r - a_r)^2$$

or equivalently

$$a_1b_1 + a_2b_2 - a_rb_r - a_s - b_s = 0 \quad .$$

This can be written by use of the 5 × 5 symmetric matrix

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{as:} \qquad {}^tASB = 0 \ .$$

Note that this is unchanged if A is replaced by Ac. Similarly for B. That is, it depends only on the homogeneous coördinates of the two points representing the circles.

1.6 It is easy to check that the condition ${}^tA\,S\,B = 0$ describes "touch" when each of the points A and B of $\mathbf{P}^4$ represents an "oriented circle". The additional cases are that either or both of A and B is an oriented line or point. For example, any two of the following touch: the line represented by ${}^t[\,0,\ 1,\ 0,\ -1,\ 2\,]$, the unit circle centered at the point (1,0) and with outward normal represented by ${}^t[\,1,\ 1,\ 0,\ 1,\ 0\,]$, and the point represented by ${}^t[\,1,\ 2,\ 0,\ 0,\ 1\,]$.

Note The condition that an "oriented circle" represented by X touches the "oriented circle" represented by A is a linear condition: ${}^tA\,S\,X = 0$, so such points X lie on a prime of $\mathbf{P}^4$.

1.7 The Lie quadric. The condition relating the homogeneous coordinates of a point representing an "oriented circle" may be written: ${}^tA\,S\,A = 0$, where S is the 5×5 matrix above. The set of points of $\mathbf{P}^4$ which satisfy the condition ${}^tX\,S\,X = 0$ is called the Lie quadric and will be denoted Ω^3. It is called a quadric since the condition is one quadratic equation in the five variables x_0, x_1, x_2, x_r, x_s . Its dimension is three, since the condition reduces the degrees of freedom by exactly one.

With one exception, the points of the Lie quadric represent "oriented circles". A point X of Ω^3 :

- represents an oriented line in the Euclidean plane
  if $x_0 = 0$ (and $x_r \neq 0$)
- represents an oriented circle in the Euclidean plane
  if $x_0 \neq 0$ and $x_r \neq 0$
- represents point in the Euclidean plane
  if $x_0 \neq 0$ and $x_r = 0$

or

- is the point ${}^t[\ 0,\ 0,\ 0,\ 0,\ 1\ ]$,
  which has no Euclidean interpretation.

In some interpretations, it is appropriate to say that this last single point represents an "ideal point". In this paper we will tacitly exclude this.

<u>1.8</u> If A is a point of Ω^3 , then the set of points X of $\mathbf{P}^4$ which satisfy the linear equation ${}^tASX = 0$, is a three-dimensional projective subspace of $\mathbf{P}^4$. The points of this prime which lie in Ω^3 represent the

"oriented circles" which touch the "oriented circle" represented by A .

The following simple observation is the basis of the algorithm. Consider three primes in P^4 :

$$ {}^{t}A\,S\,X = 0 \ , \ {}^{t}B\,S\,X = 0 \ , \ {}^{t}C\,S\,X = 0 \ . $$

These are three linear equations in five variables. If the three of the points A, B, C in $\mathbf{P}^4$ are not collinear, then these three homogeneous linear equations are independent and have a two dimensional set of solutions. Due to the "scaling", these solutions are a line in $\mathbf{P}^4$. Now, a line meets a quadric in two, one, or no points, or lies in the quadric. (Since A, B, C are not collinear, the last possibility cannot arise.) The points which lie both on the line and in the quadric Ω^3 represent "oriented circles" which touch the "oriented circles" A, B, and C .

1.9 Finally, for the sake of completeness, we mention:

Inversive separation is an invariant of Möbius geometry which extends classical inversive distance to oriented circles and oriented lines (but not points). For A and B in Ω^3, the inversive separation is given by $-{}^tA\,S\,B/A_rB_r$. If these are circles which intersect, the inversive separation equals $1 - \cos\theta$, where θ is the angle between tangents to the two circles at a common point. It vanishes when the circles touch ($\theta = 0$). In order to obtain correct interpretations of projective conditions, it is necessary to consider a point which touches an oriented circle as having any given number as the inversive separation from the circle.

Relative power is an invariant of Laguerre geometry which extends classical Steiner power to oriented circles or points (but not oriented lines). For A and B in Ω^3, the relative power is given by $-2\,{}^tA\,S\,B/A_0B_0$. If these are circles which have a common (oriented) tangent, the relative power reduces to the square of the tangential distance. It vanishes when the circles touch (tangential distance is zero). Again, in order to obtain correct interpretations of projective conditions, it is necessary to consider an oriented line which touches an oriented circle as having any given number as the relative power with respect to the circle.

With this convention: All "oriented circles" having a given inversive separation or relative power with respect to a given "oriented circle" are represented by points lying on a prime of $\mathbf{P}^4$. See [2] for details.

Here we will formulate the algorithm in terms of conditions that are spelled out simply in terms of familiar Euclidean coördinates.

2. The algorithm

1) Represent the three conditions by three linearly independent five-dimensional column vectors A , B , C according to Table 1.

2) Find any two independent choices for the column vector X so that

$$ {}^{t}A\,S\,X = 0\ ,\ {}^{t}B\,S\,X = 0\ ,\ {}^{t}C\,S\,X = 0\ . $$

Call these P and Q . Note that P and Q are two independent solutions of a system of three independent linear equations in five unknowns.

<u>Aside</u> Let K and L be any two columns vectors for which the 5-by-5 matrix $[A\ B\ C\ K\ L]$ is invertible. Then columns 4 and 5 of $({}^t[A\ B\ C\ K\ L]S)^{-1}$ will serve for P and Q . This observation permitted all calculations for this paper to be done on a pocket calculator.

3) Let u' and u'' be the roots of the quadratic equation

$${}^t(P+Qu)\,S\,(P+Qu) = 0 \quad .$$

If there are no real roots, there is no solution.

4) Multiply each of the two vectors $Z = P+Qu'$ and $P+Qu''$ by an appropriate scalar to normalize it in order to identify what "oriented circle" the point Z represents:

- In case $z_0 \neq 0$, divide by z_0 to identify an oriented circle or a point of the Euclidean plane. Obtain its center and (signed) radius from Section 1.3 or Table 1 .
- In case $z_0 = 0$ but $z_r \neq 0$, divide by $-z_r$ to identify an oriented line of the Euclidean plane. Obtain its normal and (signed) distance to the origin from Section 1.2 or Table 1.

• If both z_0 and z_r are zero, then z_1 and z_2 must be zero also, and Z is the excluded point having no interpretation in the Euclidean plane.

3. <u>An example of circles in the Euclidean plane</u>

-- with an appropriate figure -- Figure 1

To find the oriented circles having:

• tangential distance 7 from the oriented circle

$$(x_1 - 7)^2 + (x_2 - 1)^2 = (+2)^2$$

• angle $\text{Arccos}\ \frac{4}{5}$ with the oriented circle

$$(x_1 - 5)^2 + (x_2 - 3)^2 = (-5)^2$$

• centers lying on the line

$$\frac{5}{13} x_1 + \frac{12}{13} x_2 = 0$$

1) According to Table 1, these three conditions are represented by the column vectors

$$A = \begin{bmatrix} 1 \\ 7 \\ 1 \\ 2 \\ \dfrac{(7^2 + 1^2 - 2^2) - 7^2}{2} \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 5 \\ 3 \\ -5\cdot\dfrac{4}{5} \\ \dfrac{5^2 + 3^2 - 5^2}{2} \end{bmatrix}, \text{ and } \quad C = \begin{bmatrix} 0 \\ \dfrac{5}{13} \\ \dfrac{12}{13} \\ 0 \\ 0 \end{bmatrix}.$$

2) The three equations $^tA\,S\,X = 0$, ..., $^tC\,S\,X = 0$ may be written $^t[A\,B\,C]\,S\,X = 0$:

$$t\begin{bmatrix} 2 & 2 & 0 \\ 14 & 10 & 5 \\ 2 & 6 & 12 \\ 4 & -8 & 0 \\ -3 & 9 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_r \\ x_s \end{bmatrix} = 0 .$$

Here we see an advantage of homogeneous coördinates: We rescaled in order to use integers for exact calculations. (Multiply A , B , C by 2 , 2 , 13 , respectively.) This will not affect the solutions of these homogeneous equations. Such rescaling will be done routinely in the examples.

Two independent solutions to this homogeneous system of equations are:

$$P = \begin{bmatrix} 406 \\ 36 \\ -15 \\ 423 \\ 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} -34 \\ 72 \\ -30 \\ 0 \\ 423 \end{bmatrix},$$

where these are again scaled to be integers.

3) From

$${}^{t}P\,S\,P = -177408\ ,\quad {}^{t}P\,S\,Q = -168696\ ,\quad {}^{t}Q\,S\,Q = 34848$$

we obtain the quadratic equation

$$(-177408) + 2(-168696)\,u + (34848)\,u^2 = 0$$

with roots $u' = -\frac{1}{2}$ and $u'' = \frac{112}{11}$.

4) The solutions are

$$P + Q \cdot \left(-\frac{1}{2}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -\frac{1}{2} \end{bmatrix} \cdot 423$$

and

$$P + Q \cdot \left(\frac{112}{11}\right) = \begin{bmatrix} 1 \\ \frac{90}{7} \\ -\frac{75}{14} \\ \frac{99}{14} \\ 72 \end{bmatrix} \cdot \frac{47 \cdot 14}{11} .$$

From Table 1 we recognize the two oriented circles

$$x^2 + y^2 = (+1)^2$$

and

$$\left(x - \frac{90}{7}\right)^2 + \left(y + \frac{75}{14}\right)^2 = \left(\frac{99}{14}\right)^2 .$$

Figure 1 was drawn based on these calculations.

4. An example of spheres in Euclidean space

-- which can be checked by "elementary" geometry

Let four oriented spheres have their centers at $(1,1,1)$, $(1,-1,-1)$, $(-1,1,-1)$, $(-1,-1,1)$. The centers are the vertices of a regular tetrahedron lying in the cube $(\pm 1, \pm 1, \pm 1)$. Let all have the same (signed) radii be R . By elementary geometry, one finds that the spheres have their centers at the origin and radii $R \pm \sqrt{3}$. (The circumradius of the tetrahedron is $\sqrt{3}$.)

Now change the orientation of the first sphere with center at $(1,1,1)$, so it has radius $-R$.

1) According to Table 1, we use

$$A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -R \\ \frac{3-R^2}{2} \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ R \\ \frac{3-R^2}{2} \end{bmatrix} \quad C = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ R \\ \frac{3-R^2}{2} \end{bmatrix} \quad D = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ R \\ \frac{3-R^2}{2} \end{bmatrix} .$$

2) The four equations ${}^tA\,S\,X = 0$, ..., ${}^tD\,S\,X = 0$ may be written

$${}^t[A\,B\,C\,D]\,S\,X = 0 \text{ , or}$$

$$t\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -R & R & R & R \\ \frac{3-R^2}{2} & \frac{3-R^2}{2} & \frac{3-R^2}{2} & \frac{3-R^2}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * \\ * \\ * \\ * \\ * \\ * \end{bmatrix} = 0$$

Row operations on the matrix ${}^t[A\,B\,C\,D]\,S$ yields

$$\begin{bmatrix} \frac{3-R^2}{2} & 1 & 1 & 1 & -R & -1 \\ 0 & 0 & -2 & -2 & -2R & 0 \\ 0 & -2 & 0 & -2 & -2R & 0 \\ 0 & -2 & -2 & 0 & -2R & 0 \end{bmatrix} .$$

Two independent solutions to this homogeneous system of equations are:

$$P = {}^t[\ 1\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ -\frac{3-R^2}{2}\ ]$$

and

$$Q = {}^t[\ 0\ ,\ -\frac{R}{2}\ ,\ -\frac{R}{2}\ ,\ -\frac{R}{2}\ ,\ 1\ ,\ -\frac{R}{2}\ ] .$$

3) The quadratic equation is

$$(3 - R^2) + 2(R/2)\ u + (\frac{3R^2 - 4}{4})\ u^2 = 0 ,$$

and its roots are

$$u = 2\ \frac{-R \pm \sqrt{3}\ (R^2 - 2)}{3R^2 - 4} .$$

Set $\rho = \frac{-R \pm \sqrt{3}\ (R^2 - 2)}{3R^2 - 4}$. Then, the solutions are

$$P + Qu = {}^t[\ 1, -R\rho, -R\rho, -R\rho, 2\rho,\ *\ ] .$$

Here we recognize the symmetrical placement of the center of these spheres, but the radii and exact position are certainly not immediate from elementary geometry.

5. An example of spheres in Euclidean space

-- which is neither easily drawn nor easily checked by "elementary" geometry -- to illustrate the power of this approach.

To find all oriented spheres having:

- tangential distance 7 from the oriented sphere:

$$(x - 7)^2 + (y - 1)^2 + z^2 = (+2)^2$$

- angle Arccos $\frac{4}{5}$ with the oriented sphere:

$$(x - 5)^2 + y^2 + (z - 3)^2 = (-5)^2$$

- centers on the plane: $3x + 4y + 12z = 0$
- radius $+1$.

One proceeds exactly as before to obtain the two spheres

$$x^2 + y^2 + z^2 = (+1)^2$$

and

$$\left(x - \frac{648}{169}\right)^2 + \left(y + \frac{891}{169}\right)^2 + \left(z - \frac{135}{169}\right)^2 = (+1)^2 .$$

We will not attempt to draw a figure here.

Remark The first and last examples two and three dimensional examples are quite different even though the requirements on the distances, angles, and location of centers appear analogous. The three dimensional example has the additional condition for the size of the radius. These two examples were, if fact, contrived to have one solution be the unit circle or unit sphere, to have quotients of small integers for the distances, angles, and location of centers. We used the techniques described here to arrive at the appropriate given data. Since the intersection of a line with a quadric is obtained by solving a single quadratic equation, one knows that if one solution point has rational coördinates, so does the other.

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Appendix -- The Example of Section 5

The example in Secton 5 is the problem of finding all oriented spheres having:

1) tangential distance 7 from the oriented sphere
   (x - 7)^2 + (y - 1)^2 + z^2 = (+2)^2
2) angle Arccos(4/5) with the oriented sphere
   (x - 5)^2 + y^2 + (z - 3)^2 = (-5)^2
3) centers on the plane 3x + 4y + 12z = 0
4) radius +1

It will be instructive to show how we employed Lie geometry to set up this example. Note that Mathematica is used here in only a most rudimentary fashion just to check the necessary calculations. These calculations were originally carried out on an HP48S pocket calculator. When working with floating point arithmetic it is very useful to use the fact that the points which represent the geometric objects are given by vectors which can be scaled so as to replace rational number by integers -- as in the Example of Section 3. Since Mathematica handles fractions easily, this will not be employed here.

Setting up the example

The approach is to choose one sphere which will eventually be one of two solutions, and, by elementary means, find conditions regarding spheres, distances, angles, planes, and radii which are satisfied by the chosen sphere. For this example we choose the unit sphere
x^2 + y^2 + z^2 = (+1)^2
centered at the origin and with outward normal. Call this sphere x0.

First the preliminaries:

Denote by s the 6x6 matrix giving the Lie quadric (Omega)^4 in P^5 .

In[1]:=

```
s = {{ 0, 0, 0, 0, 0,-1},
     { 0, 1, 0, 0, 0, 0},
     { 0, 0, 1, 0, 0, 0},
     { 0, 0, 0, 1, 0, 0},
     { 0, 0, 0, 0,-1, 0},
     {-1, 0, 0, 0, 0, 0}} ;
```

From the 1st entry in Table 1, the sphere x0 is represented by the column vector -- or 6x1 matrix:

In[2]:=

```
x0 = {{1},{0},{0},{0},{1},{(0^2 + 0^2 + 0^2 - 1^2)/2}}
```

Out[2]=

```
                                      1
{{1}, {0}, {0}, {0}, {1}, {-(-)}}
                                      2
```

Note that Mathematica encodes a matrix as the list of its rows.

1) We look first for a sphere from which x0 will have tangential distance 7 . Consider the sphere a0

(x - 7)^2 + (y - 1)^2 + z^2 = (+2)^2 .

By drawing a figure in the xy-plane z = 0 , it is clear that the line joining the point (0,-1,0) of x0 to the point (7,-1,0) of a0 is tangent to both spheres. Both have positive radius, so that their tangential distance is 7 . The sphere a0 is represented by the column vector:

```
a0 = {{1},{7},{1},{0},{(7^2 + 1^2 + 0^2 - 2^2)/2}}
```

1)(cont'd) From the 2nd entry of Table 1, the condition that a sphere have tangential distance 7 from a0 is represented by the column vector:

In[3]:=

```
a = {{1},{7},{1},{0},{2}, \
        {(7^2 + 1^2 + 0^2 - 2^2)/2 - 7^2/2}}
```

Out[3]=

```
                                      3
{{1}, {7}, {1}, {0}, {2}, {-(-)}}
                                      2
```

Check that solution x0 satisfies this condition:

In[4]:=

```
Transpose[a].s.x0
```

Out[4]=

```
{{0}}
```

2) Next look for a sphere with which x0 will make a prescribed angle. The angle will be chosen to have a rational cosine.

Consider the spheres having center at the point

(1,0,0) + (4/5,0,3/5)*r

and passing through the point (1,0,0) -- for various non-zero values of r . By drawing a figure in the xz-plane y = 0 , it is clear that these spheres have radius -r , share with sphere x0 the point (1,0,0) , and at that point have as (inware pointing) normals vectors parallel to the vector (4/5,0,3/5) . Taking into account the orientations, the angle between sphere x0 and any one of these spheres is Arccos(4/5) = 36.87 degrees.

The parameter r may be arbitrarily chosen. Take r = 5 and denote the sphere by b0 . This sphere has center (5,0,3) and radius -5 :

(x - 5)^2 + y^2 + (z - 3)^2 = (-5)^2 .

It is represented by the column vector:

b0 = {{1},{5},{0},{3},{-5},{(5^2 + 0^2 + 3^2 - (-5)^2)/2}}

2) (cont'd) From the 4th entry of Table 1, the condition that a sphere have angle Arccos(4/5) with b0 is represented by the column vector:

In[5]:=

```
b = {{1},{5},{0},{3},{-5}*(4/5),\
      {(5^2 + 0^2 + 3^2 - (-5)^2)/2}}
```

Out[5]=

```
                                   9
{{1}, {5}, {0}, {3}, {-4}, {-}}
                                   2
```

Check that solution x0 satisfies this condition:

In[6]:=

```
Transpose[b].s.x0
```

Out[6]=

```
{{0}}
```

3) Now consider the plane through the origin given by
  3x + 4y + 12z = 0 .
Note that 3^2 + 4^2 + 12^2 = 13^2 . From the 8th entry of Table 1, this plane is represented by the column vector:
  c0 = {{0},{3/13},{4/13},{12/13},{-1},{0}} .

3)(cont'd) From the 7th entry of Table 1, the condition that a sphere be orthogonal this plane (that is, have its center on the plane) is:

In[7]:=

```
c = {{0},{3/13},{4/13},{12/13},{-1}*Cos[Pi/2],{0}}
```

Out[7]=

$$\{\{0\}, \{\frac{3}{13}\}, \{\frac{4}{13}\}, \{\frac{12}{13}\}, \{0\}, \{0\}\}$$

Check that solution x0 satisfies this condition:

In[8]:=

```
Transpose[c].s.x0
```

Out[8]=

```
{{0}}
```

4) Finally, from the 9th entry of Table 1, the condition that a sphere have radius +1 is:

In[9]:=

```
d = {{0},{0},{0},{0},{1},{-1}}
```

Out[9]=

```
{{0}, {0}, {0}, {0}, {1}, {-1}}
```

Check that solution x0 satisfies this condition:

In[10]:=

```
Transpose[d].s.x0
```

Out[10]=

```
{{0}}
```

We now have the four conditions which constitute the problem and know that sphere x0 is one solution the two solutions --- the other will have rational coordinates.

■ Carrying out the algorithm

To carry out the algorithm easily, the four column vectors which are the four conditions are placed into a 6x4 matrix m = [a b c d] . The problem reduces to solving:

```
t[a b c d].s.x = 0
t(6x4) 6x6 6x1 = 4x1
```

(t denotes transpose) for vectors x satisfying

```
tx.s.x = 0 .
```

In[11]:=

```
tm = Join[Transpose[a],Transpose[b],\
     Transpose[c],Transpose[d]]
```

Out[11]=

$$\{\{1, 7, 1, 0, 2, -(\frac{3}{2})\}, \{1, 5, 0, 3, -4, \frac{9}{2}\}, \{0, \frac{3}{13}, \frac{4}{13}, \frac{12}{13}, 0, 0\}, \{0, 0, 0, 0, 1, -1\}\}$$

Check that x0 is a solution of tm.s.x = 0 :

In[12]:=

```
tm.s.x0
```

Out[12]=

```
{{0}, {0}, {0}, {0}}
```

We proceed as though we knew neither of the two solutions.

The givens of the problem are already represented by the columns of the 6x4 matrix m = [a b c d] .

Solve tm.s.x = 0 as follows:

Adjoin to m two arbitrary columns so that the resulting 6x6 matrix is invertible. We choose the columns

In[13]:=

```
k = {{1},{0},{0},{0},{0},{0}} ;
l = {{0},{1},{0},{0},{0},{0}} ;
```

(The second matrix is "ell".)
Check that n = [a b c d k l] is invertible.
Its transpose is:

In[15]:=

```
tn = Join[tm,Transpose[k],Transpose[l]]
```

Out[15]=

$$\{\{1, 7, 1, 0, 2, -(\frac{3}{2})\}, \{1, 5, 0, 3, -4, \frac{9}{2}\},$$

$$\{0, \frac{3}{13}, \frac{4}{13}, \frac{12}{13}, 0, 0\}, \{0, 0, 0, 0, 1, -1\},$$

$$\{1, 0, 0, 0, 0, 0\}, \{0, 1, 0, 0, 0, 0\}\}$$

and its determinant is:

In[16]:=

```
Det[tn]
```

Out[16]=

$$\frac{12}{13}$$

The actual value will not be needed.

Columns 5 and 6 of the inverse of tn.s will be solutions to tm.s.x = 0 .

In[17]:=

```
z = Inverse[tn.s]
```

Out[17]=

$$\{\{-1, -1, \frac{13}{4}, -2, 2, \frac{45}{4}\}, \{0, 0, 0, 0, 0, 1\},$$

$$\{\frac{1}{2}, -(\frac{1}{2}), \frac{13}{8}, -3, 0, -(\frac{11}{8})\}, \{-(\frac{1}{6}), \frac{1}{6}, \frac{13}{24}, 1, 0, \frac{5}{24}\},$$

$$\{-1, -1, \frac{13}{4}, -3, 2, \frac{45}{4}\}, \{0, 0, 0, 0, -1, 0\}\}$$

To extract the columns, transpose the matrix
and extract the 5th an 6th rows -- as vectors -- :

In[18]:=

```
p = z[[{1,2,3,4,5,6},{5}]]
q = z[[{1,2,3,4,5,6},{6}]]
```

Out[18]=

{{2}, {0}, {0}, {0}, {2}, {-1}}

Out[19]=

$$\{\{\frac{45}{4}\}, \{1\}, \{-(\frac{11}{8})\}, \{\frac{5}{24}\}, \{\frac{45}{4}\}, \{0\}\}$$

Check:

In[20]:=

```
tm.s.p
tm.s.q
```

Out[20]=

{{0}, {0}, {0}, {0}}

Out[21]=

{{0}, {0}, {0}, {0}}

■ Description of the solution

Parameterize the line in P^5 joining p and q
as p + q*u (u real parameter). Then
Transpose(p + q*u).s.(p + q*u)
will be a quadratic polynomial in u .

In[22]:=

```
poly = Transpose[p + q*u].s.(p + q*u) ;
Expand[%]
```

Out[23]=

```
{{(-45*u)/2 - (35605*u^2)/288}}
```

```
{{(-45*u)/2 - (35605*u^2)/288}}
```

The roots of this polynomial are:

In[24]:=

```
roots = Solve[poly == 0 ,u]
```

Out[24]=

$$\{\{u \rightarrow -(\frac{1296}{7121})\}, \{u \rightarrow 0\}\}$$

The desired solutions will be represented by points given by the column vectors

In[25]:=

```
w1 = p + q*u /. roots[[1]]
w2 = p + q*u /. roots[[2]]
```

Out[25]=

$$\{\{-(\frac{338}{7121})\}, \{-(\frac{1296}{7121})\}, \{\frac{1782}{7121}\}, \{-(\frac{270}{7121})\}, \{-(\frac{338}{7121})\}, \{-1\}\}$$

Out[26]=

```
{{2}, {0}, {0}, {0}, {2}, {-1}}
```

Normalize these column vectors so that the first entry (the first entry of the first (one element) list) is one.

In[27]:=

```
w1/w1[[1]][[1]]
w2/w2[[1]][[1]]
```

Out[27]=

$$\{\{1\}, \{\frac{648}{169}\}, \{-(\frac{891}{169})\}, \{\frac{135}{169}\}, \{1\}, \{\frac{7121}{338}\}\}$$

Out[28]=

```
                                      1
{{1}, {0}, {0}, {0}, {1}, {-(-)}}
                                      2
```

From this we read off the two solutions. The second solution is the sphere x0 with which we started, the other solution is also a sphere. We read off its center and radius (and the Steiner power) to find:

```
(x - 648/169)^2 + (y + 891)^2 + (z - 135/169)^2 = (+1)^2
```

Check:

In[29]:=

```
tm.s.w1
tm.s.w2
Transpose[w1].s.w1
Transpose[w2].s.w2
```

Out[29]=

```
{{0}, {0}, {0}, {0}}
```

Out[30]=

```
{{0}, {0}, {0}, {0}}
```

Out[31]=

```
{{0}}
```

Out[32]=

```
{{0}}
```

Summary of the Algorithm

1) Represent three linear conditions by column vectors A , B , C according to Table 1.
2) Find any two independent column vectors $X = P$ and $X = Q$ so that ${}^{t}ASX = 0$, ${}^{t}BSX = 0$, ${}^{t}CSX = 0$.
3) Let u' and u" be the roots of the quadratic equation ${}^{t}(P+Qu)S(P+Qu) = 0$.
4) Multiply each of the two vectors $P + Q u'$ and $P + Q u''$ by an appropriate scalar to normalize them to ${}^{t}[\,1,\ *,\ *,\ *,\ *\,]$ or ${}^{t}[\,0,\ *,\ *,\ -1,\ *\,]$.
5) These represent the two solution: each an oriented circle (or point) or an oriented line according to Table 1.

TABLE 1

Given data	Column vector A for linear condition ${}^{t}ASX = 0$	Condition satisfied by those column vectors X which repesent:
Oriented circle with center (a_1,a_2) and signed radius $a_r \neq 0$ or point (a_1,a_2) $(a_r = 0)$	${}^{t}[\ 1,\ a_1,\ a_2,\ a_r,\ a_s\ ]$	Oriented circle or point touching the given circle or meeting the given point - or - oriented line touching the given circle or meeting the given point CLASSICAL CONDITION
Oriented circle or point $(a_r = 0)$ - and - tangential distance d (oriented with sign)	${}^{t}[\ 1,\ a_1,\ a_2,\ a_r,\ a_s - \frac{1}{2}d^2\ ]$	Oriented circle or point having tangential distance d to the given circle or point - or - oriented line touching the given circle or meeting the given point

Oriented circle or point ($a_r = 0$) - and - relative power y	${}^t[\ 1,\ a_1,\ a_2,\ a_r,\ a_s - \frac{1}{2}y\ ]$	Oriented circle or point having relative power y to the given circle or point - or - oriented line touching the given circle or meeting the given point
Oriented circle with center (a_1, a_2) - and - signed angle θ	${}^t[\ 1,\ a_1,\ a_2,\ a_r \cos\theta,\ a_s\ ]$	Oriented circle or oriented line making signed angle θ with the given circle - or - point on the given circle
Oriented circle with center (a_1, a_2) and signed radius $a_r \neq 0$ - and - inversive separation z	${}^t[\ 1,\ a_1,\ a_2,\ a_r(1-z),\ a_s\ ]$	Oriented circle or oriented line having inversive separation z with respect to the given circle - or - point on the given circle
Oriented line with normal (n_1, n_2) and signed distance n_s - and - signed angle θ	${}^t[\ 0,\ n_1,\ n_2,\ -\cos\theta,\ n_s\ ]$	Oriented circle or oriented line making signed angle θ with the given line - or - point on the given line

Oriented line with normal (n_1, n_2) - and - signed distance n_s	${}^t[\ 0,\ n_1,\ n_2,\ 0,\ n_s\ ]$	Oriented circle with center on the given line - or - point on the given line - or - oriented line perpendicular to the given line
Oriented line with normal (n_1, n_2) - and - signed distance n_s	${}^t[\ 0,\ n_1,\ n_2,\ -1,\ n_s\ ]$	Oriented circle touching the given line - or - oriented line parallel to the given line in the same direction - or - point on the given line
Signed radius R	${}^t[\ 0,\ 0,\ 0,\ 1,\ -R\ ]$	Oriented circle of signed signed radius $R \neq 0$ - or - point $(R = 0)$
	${}^t[\ 1,\ 0,\ 0,\ 0,\ 0\ ]$	Oriented line

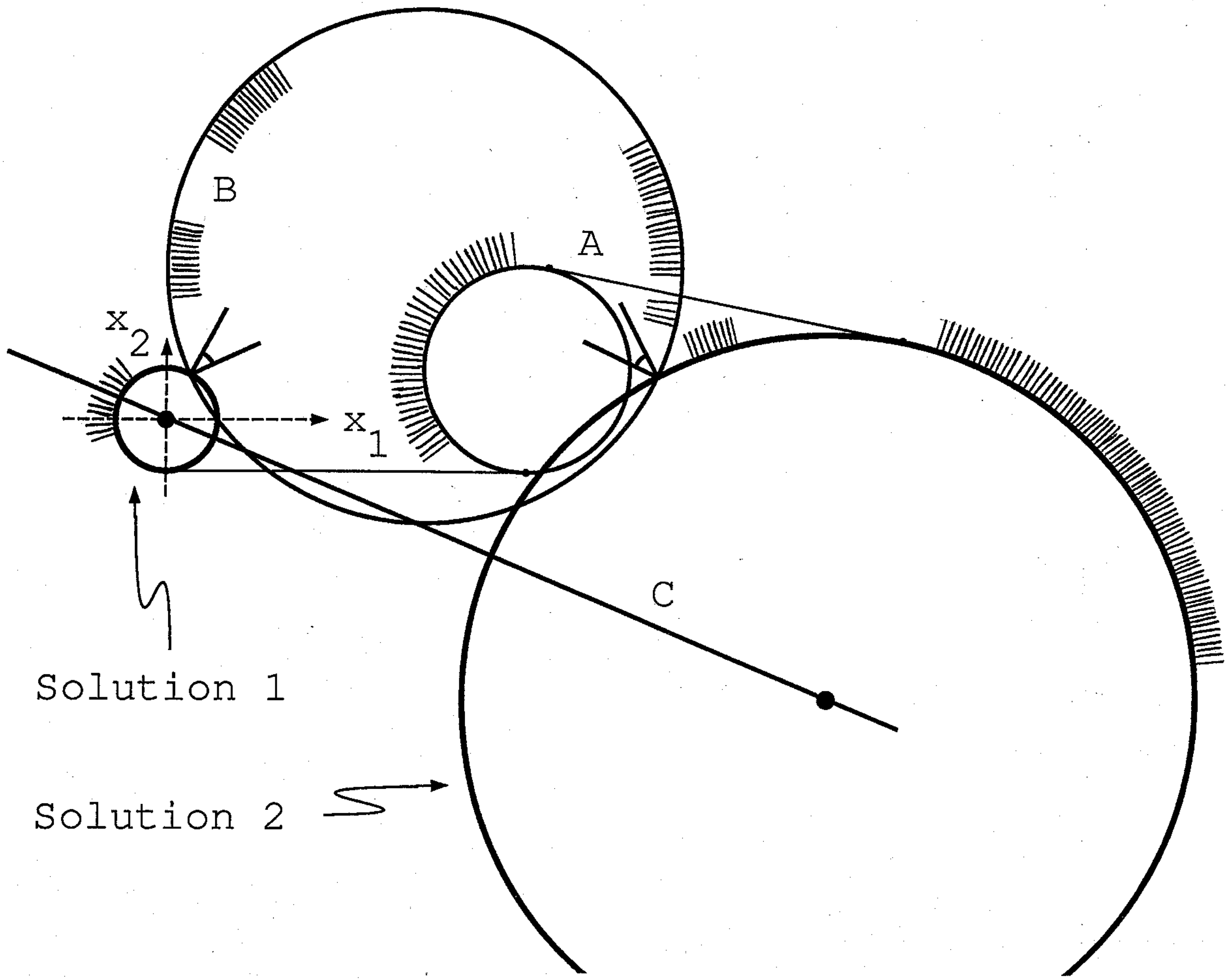
Notes for Table 1

- a_r denotes the radius of a given oriented circle with a positive sign for an outward unit normal field. a_r is zero for a point.
- $2a_s$ denotes the classical Steiner power of the origin with respect to the circle.
- In entries where a_s appears: $a_1^2 + a_2^2 - a_r^2 - 2a_s = 0$.
- d denotes the tangential distance as measured on an oriented line which touches two oriented circles (or points).
- y denotes relative power of a pair of oriented circles (or points) - an extension of classical Steiner power.
  It reduces to d^2 in case there is a common oriented tangent.
- θ denotes the angle between oriented circles (or lines) which intersect.
- z denotes inversive separation
  - an extension of classical inversive distance of circles.
  It reduces to $1 - \cos\theta$ when the circles intersect.
- (n_1, n_2) is the unit normal to an oriented line.
- n_s denotes the signed distance as measured from the origin to the line, with positive being in the direction of the unit normal.

* Column one describes, in Euclidean terms, what is the data of the condition.
* Column two is a linear condition, written as a column vector normalized for easy recognition
* Column three describes explicitly, in Euclidean terms, which "oriented circles" satisfy the linear condition.

<u>Remark</u> With the evident changes, this table is valid for other dimensions. Especially, for <u>Euclidean space</u>, one need only:

* Change 'oriented circle' to 'oriented sphere' by replacing the center (a_1,a_2) by the center (a_1,a_2,a_3) .
  Similarly for 'oriented planes'.
* Into the matrix S between the present third and fourth rows, insert an additional column of zeros and row of zeros, and insert a one on their diagonal entry.
* Note that four rather than three conditions are needed to determine the solutions.


B
A
x2
x1
C
Solution 1
Solution 2

List of symbols for the typesetter

Uppercase Latin:

A , B , C , D

A_0 , B_0 , A_r , B_r

P , Q , R , S , X

Note: Subscripts are numbers 0 , 1 , 2 and l.c. Latin i , r , s

Lower case Latin:

a_1 , a_2 , a_r , a_s

b_1 , b_2 , b_r , b_s

c , d

n_1 , n_2 , n_s

p_0 , p_1 , p_2 , p_i , p_r , p_s

u , u' , u''

x_0 , x_1 , x_2 , x_r , x_s

z_0 , z_1 , z_2 , z_r

x , y , z

Bold uppercase Latin P : $\mathbf{P}$

$\mathbf{P}^4$

<u>List of symbols</u> for the typesetter

Script ℓ (ell) to avoid confusion of 1(one) and l(ell):

ℓ_0 , ℓ_1 , ℓ_2 , ℓ_i , ℓ_r , ℓ_s

Uppercase Greek: Ω

Ω^3

Lower case Greek:

θ theta, ρ rho

Mathematical symbols:

cos cosine

$\sqrt{\ }$ square root

[ ] square brackets

${}^t[\ ]$, ${}^tA$, ${}^tB$, ${}^tC$, ${}^tD$ raised lowercase Latin t

* star - not a superscript