6. Non-Euclidean Plane Geometry

Ref. Text, Ch. II.

5. Measuring distances: Hilbert's Axioms, but unlike Playfair's Axiom refined by NE Axiom.

Measuring distances. If we fix a "unit segment" \( AB = r \), then we can measure any other segment by extending, sub-dividing, and dividing segments. We also have meaning for \( r \) and \( AB \) using Dedekind's Axiom. Call \( r \) the length of \( AB \). \( AB = A'B' \) can be understood as \( AB \) congruent to \( A'B' \) or \( AB \) and \( A'B' \) have equal lengths, without ambiguity.

On a curve - circle, hyperbola, or super-elliptic curve - lengths of an arc \( AB \) is measured by the supremum of the sum of shortest segments between finitely many adjacent points on the arc.

Note: (1) On congruent curves, \( AB = A'B' \) can be understood as \( AB \) congruent to \( A'B' \) - in the sense that there is a one-to-one correspondence between points so that drawn joining corresponding points are congruent - or \( AB \) and \( A'B' \) have the same length, without ambiguity.

a) On some curves, if \( B \) is between \( A \) and \( C \), then \( AB + BC = AC \).

b) On congruent curves:
   \( AB > A'B' \) if \( AB > A'B' \),
   \( AB = A'B' \) if \( AB = A'B' \).
   (Restrict case of curves to less than a semi-circle.)

For proof: Consider the fig. at right, \( AC = A'B' \).
Preliminary remark.

Let $A = P_0, P_1, P_2, ..., P_{n-1}, P_n = B$ be a convex polygonal line in a triangle $ABC$, e.g., each angle $< \theta$. Then

$$\sum_{i=1}^{n} P_i P_{i+1} < AC + CB.$$ 

The proof is by induction on $n$ from $\Delta AP_{n-1}C'$, where $C'$ is another $BP_{n-1}$ formed outside $AC$, and using the triangle inequality: inductive assumption $\sum_{i=1}^{n-1} P_i P_{i+1} < AC' + C'P_{n-1}$. Then

$$\sum_{i=1}^{n} P_i P_{i+1} < AC' + C'P_{n-1} + P_{n-1}B = AC' + C'B$$

$$= AC - CC' + C'B < AC - CC' + CC' + CB = AC + CB.$$

Remark. Any arc of a hexagon is a finite length.

Sketch of proof.

1) Suppose the tangent to arc $AB$ at $A$ and $B$ meets at $C$. $AB$ lies outside $\Delta ABC$.

Any polygonal line $ABC$ is convex and lies in $\Delta ABC$. The length of this line is $< AC + CB$.

The supremum of $\sum_{n=1}^{\infty} AB < AC + CB$.

2) If $AB$ is an arc such that the angle of parallelism for $\frac{1}{2}$ chord $AB$ is $> \frac{\pi}{2}$, then the tangent at $A$ and $B$ must:

- $\angle MHA > \frac{\pi}{2}$ if $\angle CMA < \frac{\pi}{2}$, $\angle AC$ exists $\angle BMA$ extended.

- Any hexagon are may be divided into finitely many arcs satisfying 2).
Call a line of the family of parallel lines defining a hexagon a major of the hexagon. Call two hexagonal concentric if they are defined by the same pencil.

Lemma. On two concentric hexagons, let one 
\( \overline{AB} \) and \( \overline{A'B'} \) be cut off by two radii in the former in the direction of the major. Then: 
\[ \overline{AA'} = \overline{BB'} \quad \text{and} \quad \overline{AB} > \overline{A'B'} \]

Proof. Since \( \overline{AA'} \) and \( \overline{BB'} \) are parallel lines, 
\[ \overline{AA'} = \overline{BB'} \]
1) Let \( \overline{CD} \) be the line of symmetry for the parallel lines \( \overline{AA'} \) and \( \overline{BB'} \). Then \( \overline{AB} \) and \( \overline{A'B'} \) are parallel to \( \overline{CC'} \) at their midpoints. Since the parallels \( \overline{AA'} \) and \( \overline{CC'} \) converge in the direction of the major, \( \overline{AC'} < \overline{AC} \). Thus \( \overline{AB} < \overline{A'B'} \).

Call \( \overline{AA'} = \overline{BB'} \) show the distance between the concentric hexagons.

Theorem. The ratio between the sides cut off on two concentric hexagons by two radii depends only on the distance between the hexagons.

Note: We must have the ratio is independent of the lengths of the sides and the location on the two hexagons. This is ensured by:

\[ \frac{\overline{AB}}{\overline{A'B'}} = \frac{\overline{CD}}{\overline{C'D'}} \]

Given: \( \overline{AA'} = \overline{CC'} \).

To show:
\[ \frac{\overline{AB}}{\overline{A'B'}} = \frac{\overline{CD}}{\overline{C'D'}}. \]
Proof 1: Case \( CD = AB \).

Let \( MM' \) and \( NN' \) be lines of symmetry of \( A'B', BB' \) and \( CC', DD' \), resp. Then \( M' \) is midpoint of \( A'B' \), \( N' \) of \( C'D' \), \( N \) of \( CD \), \( C' \) of \( C'D' \), \( B \) of \( B'B \), \( A \) of \( A'A \), \( M \) of \( MM' = NN' \). Hence \( \triangle A'MM' \cong \triangle C'NN' \) by SAS.

Using angles of paralellism and subtracting equal angles, \( \angle A'A' = \angle C'C' \) by SAS. Obtain \( C'N' = A'M' \) and \( C'N' + NN' \). Similarly \( D'N' = B'M' \) and \( D'N' + NN' \).

\( N' \) is midpoint of \( C'D' \). \( C'O' = A'B' \) so \( C'O' = A'B' \).

1) Case \( CD \) and \( AB \) are commensurable:

\( AB = \overrightarrow{A}_0B_0 \), \( CD = \overrightarrow{C}_0D_0 \), \( m \) and \( n \) integers, \( \overrightarrow{C}_0D_0 = \overrightarrow{A}_0B_0 \).

All the "equivalents" between streams are "commensurable".

by the first case, \( A'B' = \overrightarrow{A}_0B_0 \), \( C'D' = \overrightarrow{C}_0D_0 \), \( \overrightarrow{A}_0B_0 = \overrightarrow{C}_0D_0 \). Hence \( \overrightarrow{A}_0B_0 : \overrightarrow{A}'B' = \overrightarrow{C}_0D_0 : C'D' = C'B : C'O' \).

2) Case \( CD \) and \( AB \) are incommensurable.

Follows from second case by subtracting parallels.

3) \( AB \).

Cancelling. There is a positive constant \( k \) such that if \( AB \) and \( A'B' \) are and out of one line or two lines being equal by two means, the latter in the direction of parallelism, then

\( \overrightarrow{A}'B' = \overrightarrow{AB} = \overrightarrow{A}_0B_0 \),

where \( k \) in the direction between the lines.
Proof \( \overrightarrow{A'B'} / \overrightarrow{AB} \) depends only on \( x \), 
case \( x = \varphi(x) \).
\( \varphi(x) \) is a decreasing
functions of \( x \). For \( \overrightarrow{A'B'} \) not off by the same ratio,
\( \overrightarrow{A''B''} / \overrightarrow{A'B'} = \varphi(y) \). Thus
\[
\frac{\overrightarrow{A''B''}}{\overrightarrow{A'B'}} = \frac{\overrightarrow{A'B'}}{\overrightarrow{A'B'}} \frac{\overrightarrow{A'B'}}{\overrightarrow{A'B'}} = \varphi(x+y) = \varphi(x) \varphi(y).
\]
The only solution of this equation is \( \varphi(x) = e^{-\frac{x}{h}} \).
\( \log \varphi(x+y) = \log \varphi(x) + \log \varphi(y) \), so
\( \log \varphi(x) = x \log \varphi(1) \) for \( x \) constant. Since \( \log \varphi(1) \)
is decreasing, \( \varphi \) is known for \( x > 0 \) (continuity).
Decreasing means \( \log \varphi(1) < 0 \), so \( \log \varphi(1) = -\frac{1}{h} \), \( h > 0 \).
Thus \( \log \varphi(x) = -\frac{x}{h} \). \( \varphi \) is decreasing.

Remember \( k \) is independent only on the choice of unit segment UV.

1) \( k \) is the reciprocal of the radius of the NE plane.

Problem 14: A construction for \( k \).

Let \( \overrightarrow{PQ} \) be a vector of a horizontal triangle \( A \).
Let \( \overrightarrow{AC} / \overrightarrow{AB} = e \). Let the equivalence vector unit
\( \overrightarrow{l} \) or \( \overrightarrow{AC} \) and the vector triangle \( \overrightarrow{B} \) in \( P \).
If \( \overrightarrow{PQ} \) and \( \overrightarrow{CR} + \overrightarrow{l} \), then \( \overrightarrow{QR} = k \).

Since \( l \) and \( CR \) are linear, horizontal triangle \( P \).
\( \overrightarrow{PQ} = \overrightarrow{CR} \) so \( \overrightarrow{SP} = \overrightarrow{AC} \) and \( \overrightarrow{AB} / \overrightarrow{SP} = \overrightarrow{AB} / \overrightarrow{AC} = e^{-1} \). Thus
\( e^{-\frac{QR}{h}} \), so \( QR = k \). (QR = SA)
Formulas for Trajectories

Preliminary remark. Given a point A on a loxodrome, there is a unique merid of the loxodrome parallel to the tangent to the loxodrome at A.

On a ray from A making angle $\frac{\pi}{2}$ with merid $AA'$, choose $O$ so $\frac{\pi}{2}$ is a half of $\pi$ for segment $AO$.

$OO' II AA'$ is required.

The radius for $O'O$ it is parallel to the line $AA'$ at A.

Let $\sigma$ denote the length of the arc cut off on the loxodrome through $A$ by $\overrightarrow{AA'}$ and $\overrightarrow{O'O}$.

$\sigma$ is unique since it is the arc subtended by a unique chord, line $2A0$.

$\sigma$ is another constant.

Later! We will see $\sigma = k$.

Formulas. Let $\overrightarrow{AA'}$ and $\overrightarrow{BB'}$ cut of arc $AB < \sigma$ on a loxodrome.

Let $B'B'$ be drawn from $B$ parallel to the meridian $AA'$ at $A$ in $P$. Then

$\hat{AB} = \sigma$ touch $\frac{AP}{p}$ and $\frac{BP}{p} = \cos \frac{\theta}{2}$.

Proof. Construction of the figure.

Since $AB < \sigma$, the tangent line to the loxodrome are at $A$ meets $B'B'$ parallel in $P$. $PA \perp AA'$. 
i) Produce $B'B$ from $P$ to $L$ so that $PL = PA$. The perpendicular to $B'L$ at $L$ is parallel to $AP$ produced — by $PL = PA$ and equal vertical angles of parallelism at $P$. Let $L$ be parallel to both sides of the angle at $B$. Extend $AB$ to meet $L$ and construct a concurrent line of the straight $L$ meeting $L$.

ii) Let $M$ on $PB'$ so that $PM = PA$. The perpendicular to $PB'$ at $M$ is parallel to $PA$ produced — by $PM = PA$ and equal vertical angles of parallelism at $P$. Let $M$ be parallel to both sides of the angle at $B$. Extend $AB$ to meet $M$ and construct a concurrent line of the straight $M$ meeting $M$. 
2) From the concurrent forces meeting
\[ \sigma - \sigma_{AB} = \sigma_{e} \]
\[ \sigma + \frac{AB}{h} = \sigma_{e} \]

From the concurrent forces meeting
\[ \sigma + \frac{AB}{h} = \sigma_{e} \]
\[ \sigma - \frac{AP + BP}{h} = \sigma_{e} \]
\[ \sigma - \frac{AP - BP}{h} = \sigma_{e} \]

Add:
\[ \sigma = \sigma_{e} \left( \frac{AP}{h} + \frac{BP}{h} \right) = \sigma_{e} \frac{AP -BP}{h} \]
\[ \sigma = \frac{BP}{h} \cosh \frac{AP}{h} \]

Subtract from second:
\[ 2 \sigma_{AB} = \sigma_{e} \left( \frac{AP}{h} - \frac{BP}{h} \right) = \sigma_{e} \frac{AP - BP}{h} \]
\[ \sigma_{AB} = \sigma_{e} \frac{\sinh \frac{AP}{h}}{h} = \sigma_{e} \frac{\sinh \frac{AP}{h}}{h} = \sigma_{e} \tanh \frac{AP}{h} \]

Note: Cf. Test §67, pp. 136-138.

Corollary: The constants \( \sigma \) and \( h \) are equal: \( \sigma = h \).

Proof: Construct on opposite side of \( PB' \) a congruent figure of the above: \( AP = CP \), \( AB = CB' \). From the proof thatBrowse more
Lines are finite
Length: \( \overline{AC} < \overline{AP} + \overline{PC} \)
So: \( \overline{AB} < \overline{AP} \).
Let $\overline{EF}$ be $\overline{AA'}$ or $\overline{F}$. Then $\overline{AB} > \overline{AB} > \overline{FB}$ to 
$\overline{FB} < \overline{AB} < \overline{AP}$ and $\frac{\overline{FB}}{\overline{AP}} < 1$. 

As $\overline{AP}$ tends to $0$, then $\lim \frac{\overline{FB}}{\overline{AP}} = 1$. We have $\lim \frac{\overline{AB}}{\overline{AP}} = 1$.

For any $\theta$, 
$\overline{AP} = r - \tan \theta \overline{AP}$

gives 
$\frac{\overline{AB}}{\overline{AP}} = \frac{\overline{AP}}{r} - \frac{\overline{AP}}{\overline{AP}} = \frac{\overline{AP}}{r}$.

Letting $\overline{AP}$ go to $0$, we obtain $1 = \frac{\overline{AB}}{\overline{AP}}$. 

**Problem:** Text. p. 140 #5. If $s$ is the length of the arc of a lunette subtended by 
a chord of length $l$, show that $s = 2l \sin \left( \frac{l}{2r} \right)$.

**Solution:**

$$\overline{OP} = \overline{AB} \cos \frac{\overline{AP}}{r}$$

$$\overline{AB} = r - \tan \theta \overline{AP}$$

From $\cos \frac{\overline{AP}}{r} = \cos \frac{\overline{AP}}{r}$,

$$\overline{OP} = r - \tan \theta \overline{AP} \cos \frac{\overline{AP}}{r}$$

$$= s \sin \frac{\overline{AP}}{r}$$

Hence use $\overline{OP} = \frac{s}{2}$, $\overline{AP} = \frac{r}{2}$.

**Right Triangles**

Let $\triangle ABC$ be a plane right triangle such that: 
acute angle $\alpha = \angle A$, $\beta = \angle B$ and right angle $\gamma = \angle C$. 
Legs $\overline{a} = \overline{BC}$, $\overline{b} = \overline{AC}$, and hypotenuse $\overline{c} = \overline{AB}$.

Let $\overline{AA'}$ be plane $\overline{ABC}$ or $A$ and let $\overline{BB'}$ be $\overline{AA'}$. 
These parallel lines determine a transverse 
plane to plane $\overline{ABC}$ at $A$. Plane $\overline{AA'BB'}$ intersects the transverse in $\overline{AB}$, between $\overline{AA'}$ and $BB'$. 
Likewise planes $\overline{AA'CC'}$ and $\overline{BB'CC'}$. Obtain 
triangle $\triangle AB'C$, on the transverse, starting 
right angle $\angle BAC$, $\overline{BB'}$ at $B$, draw from $\overline{AA'}$ to 
plane $\overline{ABC}$. Name $\overline{AA'}$. Also plane $\overline{ABC}$. 

plane AA'CC' I plane ABC. BC is perpendicular to their line of intersection CA so BC⊥plane AA'CC'.

There is a plane BB'CC' I plane AA'CC' and different angle CC' is a right angle. ΔABC, c₁ is a right triangle whose sides: \( a = a₁ \), \( c₁ = c \), \( b = b₁ \) be its sides, \( c₁ = AB \) as hypotenuse. ΔABC, c₁ is the projection of ΔABC onto the transversal.

Let \( B_2C \) be a line that is parallel to \( B_1C \) in plane BB'CC' \( a₂ = B_2C \) is its length, and \( d = CC' = B_2C \), the distance between the two lines.

From the first of the projection formulas:

\[
\begin{align*}
\theta &= \sigma \tan \frac{b}{h} \\
\theta &= \sigma \tan \frac{b}{h} \quad \therefore \quad c₁ = \sigma \tan \frac{c}{h}
\end{align*}
\]

and \( a₂ = \sigma \tan \frac{a}{h} \).

From the second of the projection formulas we get \( c = \frac{c₁}{\sigma} \) = \( \cosh \frac{c}{h} \). From \( a₁ = a₂ e^{-\frac{c}{h}} \) we obtain

\[
\begin{align*}
a₁ &= \sigma \frac{\tan \frac{a}{h}}{\cosh \frac{c}{h}}
\end{align*}
\]

These formulas for \( a₁, b, c \), allow one to obtain NE trigonometry formulas from Euclidean trigonometry.
By Pythagorean theorem,

\[
cosh \frac{c}{h} = \cosh \frac{a}{h} \cosh \frac{b}{h}
\]

**Proof 1)** Note from \( \cosh^2 x - \sinh^2 x = 1 \), obtain

\[
1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \quad \cosh^2 x = \frac{1}{1 - \tanh^2 x}.
\]

2) From Euclidean Pythagorean Theorem on the hypotenuse:

\[
c_1^2 = a_1^2 + b_1^2.
\]

So:

\[
\tanh^2 \frac{c}{h} = \tanh^2 \frac{a}{h} + \tanh^2 \frac{b}{h} \quad \text{(after division)}
\]

\[
= \tanh^2 \frac{a}{h} \left(1 - \tanh^2 \frac{b}{h}\right) + \tanh^2 \frac{b}{h}
\]

\[
= \tanh^2 \frac{a}{h} + \tanh^2 \frac{b}{h} - \tanh^2 \frac{a}{h} \tanh^2 \frac{b}{h}
\]

\[
= 1 - \left(1 - \tanh^2 \frac{a}{h}\right)\left(1 - \tanh^2 \frac{b}{h}\right)
\]

or

\[
1 - \tanh^2 \frac{c}{h} = \left(1 - \tanh^2 \frac{a}{h}\right)\left(1 - \tanh^2 \frac{b}{h}\right).
\]

Taking reciprocals:

\[
\cosh^2 \frac{c}{h} = \cosh^2 \frac{a}{h} \cosh^2 \frac{b}{h}.
\]

Formulas relating sine, cosine, and an angle

\[
\tanh \frac{b}{h} = \tanh \frac{a}{h} \sec
\]

\[
\sin \frac{a}{h} = \sin \frac{c}{h} \sin \alpha
\]

\[
\tanh \frac{b}{h} = \sin \frac{b}{h} \sec \alpha
\]

**Proof 1)** \( b_1 = c_1 \\cos \alpha \) yields

\[
\sin \frac{b}{h} = \sin \frac{c}{h} \sec \alpha
\]

and the first formula.

1) \( c_1 = c_1 \sin \alpha \) yields

\[
\tanh \frac{b}{h} = \sin \frac{b}{h} \sec \alpha
\]
\[
\begin{align*}
\frac{\sinh \frac{a}{n}}{\cosh \frac{a}{n} \cosh \frac{b}{n}} &= \frac{\sinh \frac{c}{n}}{\cosh \frac{c}{n}} \sin x.
\end{align*}
\]

The denominators are squared by the Pythagorean Theorem, so obtain second formula.

3) \( a = b \), \( \tan x \) given

\[
0 = \frac{\tan \frac{a}{n}}{\cosh \frac{a}{n}} = \frac{\tan \frac{b}{n}}{\cosh \frac{b}{n}} \tan x
\]

and the third formula. \( \Box \)

**Formulas relating time angles and a cube.**

\[
\begin{align*}
\cos \alpha &= \cosh \frac{a}{n} \sin \beta \\
\cosh \frac{c}{n} &= \cos \alpha \cosh \beta
\end{align*}
\]

**Remark.** There are analogous formulas in spherical trigonometry, but \( E = \sin \alpha \cos \beta = \sin \beta \cos \alpha \) is not. Examine Ex. 3.0: \( \cos \alpha = \sin \beta, \cos \alpha \cosh \beta = 1 \).

**Proof.**

1) \( \tan \frac{a}{n} = \sin \frac{b}{n} \tan x \)

\[
\tan \frac{b}{n} = \sinh \frac{a}{n} \tan \beta
\]

Multiply:

\[
\frac{\sinh \frac{a}{n}}{\cosh \frac{a}{n} \cosh \frac{b}{n}} = \frac{\sinh \frac{b}{n}}{\cosh \frac{b}{n} \cosh \frac{a}{n}} \sinh \frac{a}{n} \tan \beta.
\]

By Pythagorean Theorem gives \( \frac{1}{\cosh \frac{a}{n}} = \tan \alpha \tan \beta \)

and the second formula.

2) \( \tan \frac{b}{n} = \tan \frac{c}{n} \cos \alpha \)

\[
\sin \frac{b}{n} = \sin \frac{c}{n} \sin \beta \quad \text{(second formula)}
\]

Divide:

1. \( \frac{1}{\cosh \frac{b}{n}} = \frac{1}{\cosh \frac{c}{n} \sin \beta} \)

Pythagorean Theorem gives:

\[
\frac{1}{\cosh \frac{a}{n}} = \frac{1}{\cosh \frac{b}{n} \sin \beta}
\]

and the first formula. \( \Box \)
Remark. These formulas may be summarized by
Napier’s rules as in spherical trigonometry.
One uses hyperbolic functions rather than
circular functions common for angles.

**Summary of formulas**

\[
cosh \frac{c}{h} = \cosh \frac{a}{h} \cosh \frac{b}{h}
\]

\[
cosh \frac{c}{h} = \cosh \frac{a}{h} \cosh \frac{b}{h}
\]

\[
\tanh \frac{a}{h} = \tanh \frac{b}{h} \cosh \beta
\]

\[
\sinh \frac{a}{h} = \sinh \frac{b}{h} \sin \beta
\]

\[
\tanh \frac{a}{h} = \sinh \frac{b}{h} \tan \beta
\]

\[
\cos \beta = \cosh \frac{b}{h} \sin \alpha
\]

\[
\cos \alpha = \cosh \frac{a}{h} \sin \beta
\]

*Text, p. 152.*

**General triangle**

Let \( AD = BC = D \)

(figure drawn

with \( AD \) horizontal \( BC \).

\( h = AD \), altitude,

**Law of**

\[
\cosh \frac{c}{h} = \cosh \frac{a}{h} \cosh \frac{b}{h} - \sinh \frac{a}{h} \sinh \frac{b}{h} \cos \gamma
\]

**Proof.** \[
\cosh (x-y) = \cosh x \cosh y - \sinh x \sinh y
\]

Then:

\[
\cosh \frac{c}{h} = \cosh \frac{a}{h} \cosh \frac{b}{h} - \sinh \frac{a}{h} \sinh \frac{b}{h} (1 - \tanh \frac{a}{h} \tanh \frac{b}{h})
\]

\[
= \cosh \frac{a}{h} \cosh \frac{b}{h} \left( 1 - \frac{\tanh \frac{a}{h} \tanh \frac{b}{h}}{\cosh \frac{a}{h} \cosh \frac{b}{h}} \right)
\]

\[
= \cosh \frac{a}{h} \cosh \frac{b}{h} (1 - \tanh \frac{a}{h} \tanh \frac{b}{h} \cot \gamma)
\]

\[
= \cosh \frac{a}{h} \cosh \frac{b}{h} - \sinh \frac{a}{h} \sinh \frac{b}{h} \cosh \gamma
\]

*Q.E.D.*
Law of Sines: \[
\frac{\sin \alpha}{ \text{side } a} = \frac{\sin \beta}{ \text{side } b} = \frac{\sin \gamma}{ \text{side } c}.
\]

Proof:
\[
\sin \frac{A}{2} = \sin \frac{B}{2} \sin \gamma = \sin \frac{C}{2} \sin \beta
\]
(triangle on left) (triangle on right)
\[
\Rightarrow \frac{\sin \beta}{\text{side } b} = \frac{\sin \gamma}{\text{side } c}. \text{ The others similarly.}
\]

Problems:
1. Look up the proofs of the Theorems of Menelaus and Ceva in Euclidean geometry. They are found in Exercise Text, Problems 147-148, #1, #3.


\[\text{Problem 147:}\]
If a straight line cuts the sides of triangle ABC, dividing them as a ratio segments \(a_1\) and \(a_2\), \(b_1\) and \(b_2\), \(c_1\) and \(c_2\), from that point, we have:
\[
\frac{\sin \frac{A}{2}}{\text{side } a} = \frac{\sin \frac{B}{2}}{\text{side } b} \cdot \frac{\sin \frac{C}{2}}{\text{side } c}.
\]
(The converse of the Theorem of Menelaus.)

\[\text{Problem 148:}\]
If the lines joining point P to the vertices of triangle ABC divide the sides into pairs of segments \(a_1\) and \(a_2\), \(b_1\) and \(b_2\), \(c_1\) and \(c_2\), from that point:
\[
\frac{\sin \frac{A}{2}}{\text{side } a} = \frac{\sin \frac{B}{2}}{\text{side } b} \cdot \frac{\sin \frac{C}{2}}{\text{side } c} \cdot \frac{\sin \frac{D}{2}}{\text{side } d}.
\]
(Compare with the Theorem of Ceva.)
Remark: "Sum of cosines" for angles:

\[ \cos \gamma = \frac{\cos \alpha \cdot \cos \beta + \cos \gamma}{\sin \alpha \cdot \sin \beta} \]


Relationship to Euclidean and spherical trigonometry

Euclidean Using MacLaurin series
\[ \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \]
\[ \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \]

Law of Cosines
\[ \cosh \frac{c}{R} = \cosh \frac{a}{R} \cosh \frac{b}{R} - \sinh \frac{a}{R} \sinh \frac{b}{R} \cos \frac{c}{R} \]

becomes
\[ 1 + \frac{c^2}{2R^2} + \ldots = \left( 1 + \frac{a^2}{2R^2} + \ldots \right) \left( 1 + \frac{b^2}{2R^2} + \ldots \right) \]
\[- \left( \frac{a}{R} + \ldots \right) \left( \frac{b}{R} + \ldots \right) \cos \frac{c}{R} \]

Inclined terms \( \frac{1}{4R^2} \) and higher orders are omitted. Cancel 1 and multiply by \( \frac{1}{2R^2} \):
\[ c^2 + \ldots = \left( a^2 + \frac{b^2}{R^2} + \ldots \right) - \left( \frac{a b}{R} + \ldots \right) \cos \frac{c}{R} \]

Inclined terms \( \frac{1}{4R^2} \) and higher orders are omitted. But \( R \to \infty \):
\[ c^2 = a^2 + \frac{b^2}{R^2} - \frac{a b}{R} \cos \frac{c}{R} \]

This is an Euclidean law of cosines.

Similarly: Other formulas yield Euclidean versions at \( R \to \infty \).

Interpretation: 1) \( R \to \infty \) becomes infinitely large, NE geometry becomes Euclidean geometry.
2) NE geometry is Euclidean on the small:
when lengths are small compared with \( R \),
Euclidean trigonometry formulas almost hold.

Special Using the relations
\[ \sin \frac{\pi x}{R} = \sqrt{1 - \sin^2 \frac{x}{R}} \quad \cos \frac{\pi x}{R} = \cos \frac{x}{R} \]
the spherical law of cosines for angles of \( \pi / R \)
\[ \cosh \frac{c}{R} = \cosh \frac{a}{R} \cosh \frac{b}{R} - \sinh \frac{a}{R} \sinh \frac{b}{R} \cos \frac{c}{R} \]
For $R = \frac{1}{2} k$:
\[
\cosh \frac{\theta}{2} = \cosh \frac{\theta}{2} \cosh \frac{\theta}{2} - \sinh \frac{\theta}{2} \sinh \frac{\theta}{2} \cosh \frac{\theta}{2}
\]

Similarly: Other spherical formulas yield NE version for $R = \frac{1}{2} k$.

Interpretation: NE geometry is spherical geometry on a sphere of imaginary radius $R = \frac{1}{2} k$.

Angle of parallelism formula

(Loba chernykh - Boluyan)

\[
\beta = PO
\]

\[
\Pi(\beta) = AB'PO
\]

\[
\tan \frac{\Pi(\beta)}{2} = e^{-\frac{\beta}{k}}
\]

Proof: 1) Let $Q$ on $OA'$, $a = OQ$.

Note that $\alpha \to \Pi(\beta)$ as $OQ \to 0$.

2) From

\[
\sin \beta = \frac{\cosh \frac{\theta}{2}}{\cosh \frac{\theta}{2}} < \frac{1}{1 - \cosh \frac{\theta}{2}}
\]

concluded! $\beta \to 0$ as $OQ \to 0$.

From

\[
\sin \alpha = \frac{\cosh \beta}{\cosh \frac{\theta}{2}}, \quad \cot \alpha = \frac{\tanh \frac{k}{2}}{\tanh \frac{\theta}{2}}
\]

conclude $\alpha \to 0$ as $OQ \to 0$:

\[
\sin \Pi(\beta) = \frac{1}{\cosh \frac{\theta}{2}}, \quad \cosec \Pi(\beta) = \frac{\tanh \frac{k}{2}}{\tanh \frac{\theta}{2}}
\]

3) From

\[
\tan \frac{\beta}{2} = \frac{1 - \cosec x}{\cosec \frac{x}{2}}
\]
We get
\[ \tan \frac{\pi}{2} = \frac{1 - \sin \frac{k}{2}}{\cos \frac{k}{2}} = \frac{\sin k}{\cos k} = \frac{e^{-k}}{1} = \frac{1}{k} \]

**Note:** As \( k \to \infty \), \( \tan \frac{\pi}{2} \) becomes \( \pi \) giving \( \frac{\pi}{2} = \frac{\pi}{2} \) for Euclidean geometry.

**Remark:** Lobachevsky arrived at this form of hyperbolic and circular functions. This form of the laws of cones was given:

\[ \sin \Pi(c) = \frac{\sin \Pi(c) \sin \Pi(b)}{1 - \cos \Pi(c) \cos \Pi(b)} \]

**Problem:**

a) Show \( \Pi(b) = \frac{k}{2} = \frac{\pi}{2} \) for \( \frac{k}{2} = \log (1 + k) \).

b) Derive Lobachevsky's form of laws of cones.

**Solution:**

a) \( \tan \frac{\pi}{2} = \frac{1}{\cos \frac{k}{2}} = 1 - \frac{k}{2} + \frac{1}{2!} \).

b) Use

\[ \sin \Pi(b) = \frac{1}{\cosh \frac{k}{2}} \]

which yields

\[ \sin \frac{k}{2} = \frac{\cosh \Pi(b)}{\sinh \Pi(b)} \]

\[ \cosh \frac{k}{2} = \cosh \frac{\Pi(a)}{\cosh \Pi(b)} \]

\[ \cosh \frac{k}{2} = \cosh \frac{\Pi(a)}{\cosh \Pi(b)} \]

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\[ \sinh \frac{k}{2} = \sinh \frac{\Pi(a)}{\sinh \Pi(b)} \]
Remark: Law of cosines for angles:

\[ \cos \frac{\theta}{2} = \frac{\cos \alpha + \cos \beta}{2} \]


6) Prove, for a right triangle with \( c \) as the hypotenuse:

- **Sine Formulas:**
  \( \sin \left( \frac{\pi}{2} - \alpha \right) = \sin \alpha \)
  \( \sin \left( \frac{\pi}{2} - \beta \right) = \cos \beta \)

- **Cosine Formulas:**
  \( \cos \left( \frac{\pi}{2} - \alpha \right) = \cos \alpha \)
  \( \cos \left( \frac{\pi}{2} - \beta \right) = \sin \beta \)

Remark: Cosecants of supplementary angles:

\[ \csc \left( \pi - \theta \right) = - \csc \theta \]

For example, Szmiełowicz, Foundations of Geometry, Ch. IV, p. 26, p. 334 ff.
Astronomical estimate of \( k \). (Hooke's law).

\[ S = \text{sum} \]
\[ E = \text{Earth} \]
\[ A = \text{star} \]
\[ \omega = \text{radius of Earth's orbit} \]
\[ x = \text{parallel of star} \]
\[ \alpha = \text{angle between parallel of star to BS or E and EA.} \]

**Claim:** \( \frac{k}{n} > \frac{a}{\tan x} \)

**Proof:** 1. **Arithmetic inequality:** For \( 0 < x < 1 \):

\[
\log \frac{1 + x}{1 - x} = 2 \left( x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots \right)
\]
\[
\frac{2x}{1 - x^2} = 2 \left( x + x^3 + x^5 + x^7 + \ldots \right)
\]

So,

\[
\log \frac{1 + x}{1 - x} < \frac{2x}{1 - x^2} \quad \text{for} \quad 0 < x < 1.
\]

2. **Given:** \( EA \) meets \( SA \) at \( A \), along straight lines.

\[
\frac{IT(a)}{2} > \angle AES = \frac{\pi}{2} - \alpha, \quad \text{and} \quad \frac{IT(a)}{2} > \frac{\pi}{2} - \frac{\alpha}{2}.
\]

Then,

\[
e^{-\frac{a}{k}} = \tan \frac{IT(a)}{2} > \tan \left( \frac{\pi}{2} - \frac{\alpha}{2} \right) = \frac{1 - \tan \frac{\alpha}{2}}{1 + \tan \frac{\alpha}{2}} = \left( 1 + \tan \frac{\alpha}{2} \right)
\]

and,

\[
e^{-\frac{a}{k}} < \frac{1 + \tan \frac{\alpha}{2}}{1 - \tan \frac{\alpha}{2}}.
\]

Using the inequality \( e^x > 1 + x \):

\[
e^{-\frac{a}{k}} < \log \frac{1 + \tan \frac{\alpha}{2}}{1 - \tan \frac{\alpha}{2}} < \frac{1 + \tan \frac{\alpha}{2}}{1 - \tan \frac{\alpha}{2}} = \tan x.
\]

**Note:**

**Numerical calculation:**

Using star Sirius (approx. 8.7 light years from Earth) with parallel \( \alpha = 1.0^\circ \) (Hooke's law's data) we have:

Using star Sirius (approx. 8.7 light years from Earth) with parallel \( \alpha = 1.0^\circ \) (Hooke's law's data) we have:
\[
\tan \alpha \times \tan \beta \times \frac{1}{2} = 1.24 \times \frac{\pi}{3600} = 6.012 \times 10^{-6}
\]
\[
\alpha = 1.496 \times 10^{-12} \text{ cm} = 92960000 \text{ miles}
\]
\[
\frac{\alpha}{\tan \alpha} = \frac{1.496 \times 10^{-13}}{1.601 \times 10^{-5}} = 3.487 \times 10^{-8} \text{ cm} = 1.546 \times 10^{-13} \text{ miles}
\]

Name:
1 light-year = 9.4605 \times 10^{12} \text{ cm} = 5.879 \times 10^{12} \text{ miles}
\[
\frac{\alpha}{\tan \alpha} = \frac{3.487 \times 10^{-12}}{9.4605 \times 10^{12}} = 15.46 \times 12 = 5.879 \times 10^{12} = 3.630 \text{ light-years}
\]

Thus: \( h > 2.63 \) light-years.

Remark: Easy to calculate: \( h > \frac{3.342}{\alpha} \text{ light-years} \)

If one finds a star with parallax \( \alpha = 0.1'' \text{ say,} \) then one can conclude \( h \geq 2000000 \text{ or} \ h > 32.6 \text{ light-years} \)

Note: For \( h = \text{Everest} = \text{star A} \), \( \alpha = \frac{d}{\tan \alpha} \),
\( h \geq d \).

From: Newton's Star Atlas

<table>
<thead>
<tr>
<th>Star</th>
<th>( \alpha )</th>
<th>( d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sirius</td>
<td>0°.371</td>
<td>9 light-years</td>
</tr>
<tr>
<td>\beta Centauri</td>
<td>0°.011</td>
<td>300</td>
</tr>
<tr>
<td>Antares</td>
<td>0°.009</td>
<td>360</td>
</tr>
</tbody>
</table>

For Sirius: \( 3.262 \times 0.371 = 1.21 \text{ light-years} \)

Note: 1'' of arc is subtended by a \( \alpha = 1 \) parsec.
1 parsec = 3.262 light-years.