6. Non-Euclidean plane trigonometry

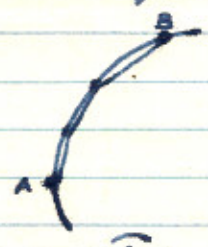
Ref. Text. Ch. V.

Standing assumptions: Hilbert's Axioms, but with Playfair's Axiom replaced by NE Axiom.

Measuring distance

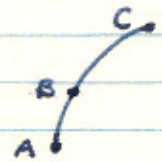
If we fix a "unit segment" UV, $AB = r\,UV$ has meaning for r rational by extending superposing and dividing segments. It also has meaning for r real using Dedekind's Axiom. Call r the length of AB. $AB = A'B'$ can be interpreted as AB congruent to $A'B'$ or AB and $A'B'$ have equal length without ambiguity.

On a curve - circle, horocycle, or equidistant curve - length of an arc $\widehat{AB}$ is measured by the supremum of the sum of chord lengths between finitely many adjacent points on the arc.



Note: 1) On congruent curves, $\widehat{AB} = \widehat{A'B'}$ can be interpreted as $\widehat{AB}$ congruent to $\widehat{A'B'}$ - in the sense that there is a one-to-one correspondence between points so that chords joining corresponding points are congruent - or $\widehat{AB}$ and $\widehat{A'B'}$ have the same length without ambiguity.

2) On one curve, if B is between A and C, then $\widehat{AB} + \widehat{BC} = \widehat{AC}$.

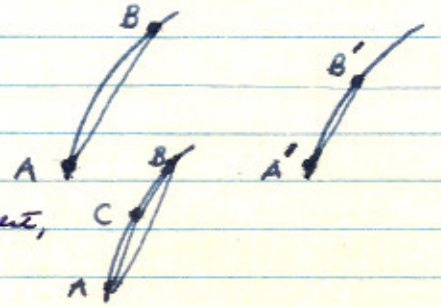


3) On congruent curves:

$\widehat{AB} > \widehat{A'B'}$ if $AB > A'B'$,

$\widehat{AB} = \widehat{A'B'}$ if $AB = A'B'$.

(Restrict arcs of curves to less than a semi-circle.)



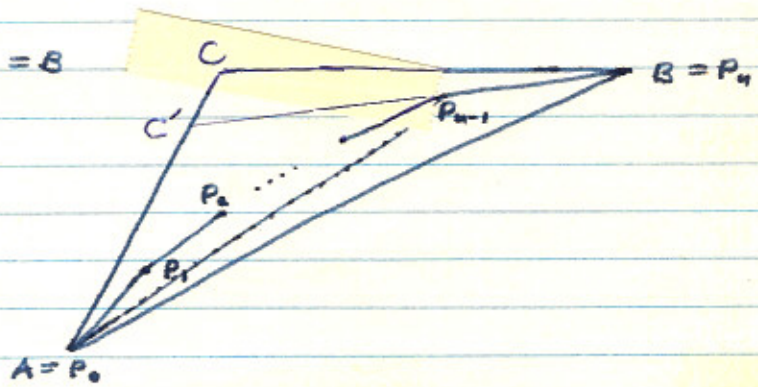
For proof: Consider fig. at right, with $AC = A'B'$.

5/7 **Horocycles**

Preliminary remark.

Let $A = P_0, P_1, P_2, \dots, P_{n-1}, P_n = B$ be a convex polygonal line in a triangle ABC s.t. each angle $\angle P_{i-1} P_i P_{i+1} < 2\rho$. Then

$$\sum_{i=1}^{n} P_{i-1} P_i < AC + CB.$$



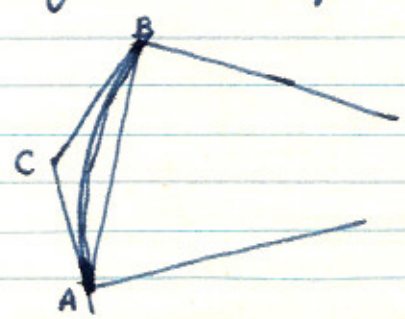
The proof is by induction on n from $\Delta A P_{n-1} C'$, where C' is where BP_{n-1} produced meets AC, and using the triangle inequality: Inductive assumption: $\sum_{i=1}^{n-1} P_{i-1} P_i < AC' + C'P_{n-1}$. Then

$$\sum_{i=1}^{n} P_{i-1} P_i < AC' + C'P_{n-1} + P_{n-1}B = AC' + C'B$$
$$= AC - CC' + C'B$$
$$< AC - CC' + CC' + CB = AC + CB.$$

Remark. Any arc of a horocycle has finite length.

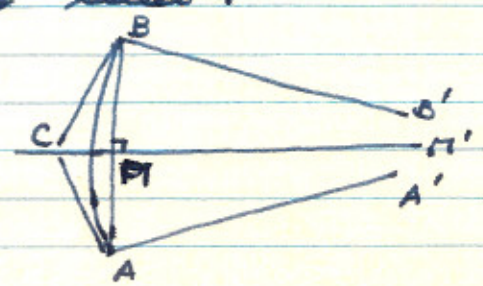
Sketch of proof.

1) Suppose the tangents to arc $\widehat{AB}$ at A and B meet at C. $\widehat{AB}$ lies interior to ΔABC. Any polygonal line A to B is convex and lies in ΔABC. The length of this line is $< AC + CB$. The supremum gives $\widehat{AB} < AC + CB$.



2) If $\widehat{AB}$ is an arc such that the angle of parallelism for $\frac{1}{2}$ chord AB is $> \frac{\rho}{2}$, then the tangents at A and B do meet:

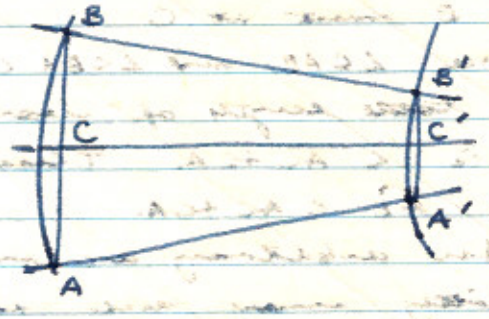
$\angle A'AM > \frac{\rho}{2}$ implies $\angle CAM < \frac{\rho}{2}$, so AC meets $M'M$ produced.



3) Any horocycle arc may be divided into finitely many arcs satisfying 2).

Call a line of the pencil of parallel lines defining a horocycle a <u>radius</u> of the horocycle. Call two horocycles <u>concentric</u> if they are defined by the same pencil.

<u>Lemma</u> On two concentric horocycles, let arcs $\widehat{AB}$ and $\widehat{A'B'}$ be cut off by two radii - the latter in the direction of parallelism. Then: $AA' = BB'$ and $\widehat{AB} > \widehat{A'B'}$.



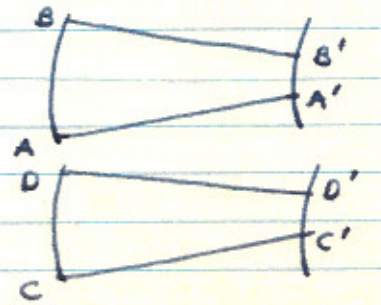
<u>Proof</u> 1) Since A and A' correspond to B and B' on parallel rays, $AA' = BB'$.

2) Let CC' be the axis of symmetry for the parallel rays AA' and BB'. Then AB and $A'B'$ are perpendicular to CC' at their midpoints. Since the parallels AA' and CC' converge in the direction of parallelism, $A'C' < AC$. Thus $A'B' < AB$, and $\widehat{A'B'} < \widehat{AB}$. QED.

Call $AA' = BB'$ above the <u>distance</u> between the concentric horocycles.

<u>Theorem</u>. The ratio between the arcs cut off on two concentric horocycles by two radii defines <u>any</u> on the distance between the horocycles.

<u>Note</u>: We must prove the ratio is independent of the lengths of the arcs <u>and</u> the location on the two horocycles. This is accomplished by:

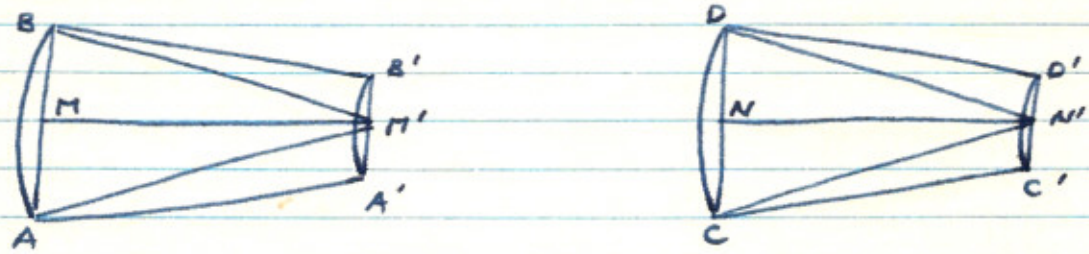


<u>Given</u>: $AA' = CC'$.

<u>To show</u>:

$$\widehat{AB} : \widehat{A'B'} = \widehat{CD} : \widehat{C'D'}.$$

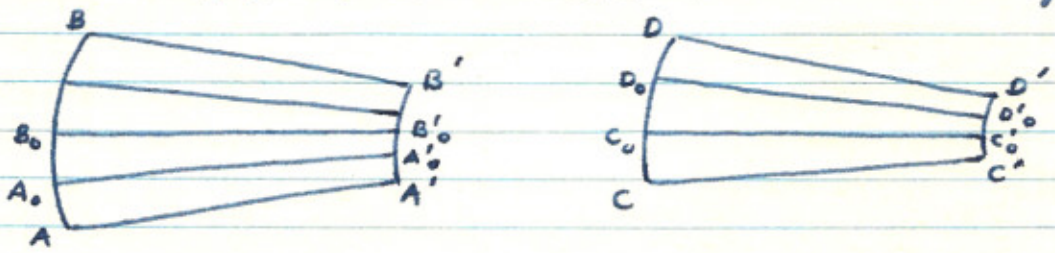
Proof 1) Case $\widehat{CD} = \widehat{AB}$.



Then $CD = AB$. Let MM' and NN' be lines of symmetry of AA', BB' and CC', DD' resp. Then M is midpoint of AB, M' of $A'B'$, N of CD. Locate N' so $MM' = NN'$. Then $\Delta AMM' \cong \Delta CNN'$ by SAS. Using angles of parallelism and subtracting equal angles, $\Delta A'AM' \cong \Delta C'CN'$ by SAS. Obtain $C'N' = A'M'$ and $C'N' \perp NN'$. Likewise $D'N' = B'M'$ and $D'N' \perp NN'$. N' is midpoint of $C'D'$. $C'D' = A'B'$ so $\widehat{C'D'} = \widehat{A'B'}$.

2) Case $\widehat{CD}$ and $\widehat{AB}$ are commensurable:

$\widehat{AB} = m\,\widehat{A_0B_0}$, $\widehat{CD} = n\,\widehat{C_0D_0}$, m and n integers, $\widehat{C_0D_0} = \widehat{A_0B_0}$.



All the "quadrilaterals" shown are "congruent", by the first case, so $\widehat{A'B'} = m\,\widehat{A'_0B'_0}$, $\widehat{C'D'} = n\,\widehat{C'_0D'_0}$ and $\widehat{A'_0B'_0} = \widehat{C'_0D'_0}$. Then

$\widehat{AB} : \widehat{A'B'} = \widehat{A_0B_0} : \widehat{A'_0B'_0} = \widehat{C_0D_0} : \widehat{C'_0D'_0} = \widehat{CD} : \widehat{C'D'}$.

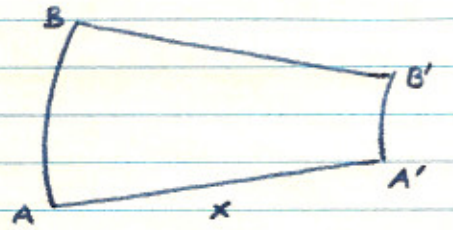
3) Case $\widehat{CD}$ and $\widehat{AB}$ are incommensurable.

Follows from second case by ~~[illegible]~~ Dedekind's Ax com. QED.

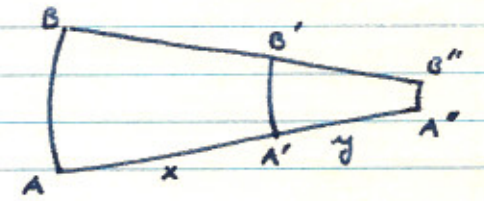
Corollary There is a positive constant h such that if $\widehat{AB}$ and $\widehat{A'B'}$ are arcs cut off on two concentric horocycles by two radii, the latter in the direction of parallelism, then

$$\widehat{A'B'} = \widehat{AB}\, e^{-\frac{x}{h}}$$

where x is the distance between the horocycles.



$$\widehat{A'B'} = \widehat{AB}\, e^{-\frac{x}{h}}$$



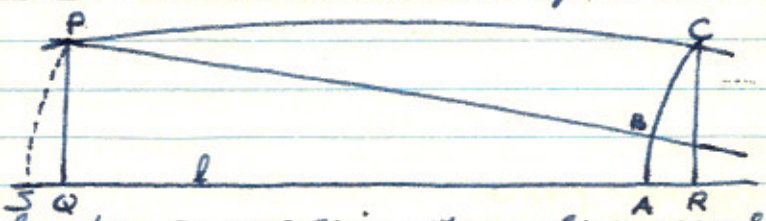
<u>Proof</u> $\widehat{A'B'}/\widehat{AB}$ depends only on x, call it $\varphi(x)$.
$\varphi(x)$ is a decreasing function of x. For $\widehat{A''B''}$ cut off by the same radii, $\widehat{A''B''}/\widehat{A'B'} = \varphi(y)$. Then

$$\frac{\widehat{A''B''}}{\widehat{AB}} = \frac{\widehat{A'B'}}{\widehat{AB}} \cdot \frac{\widehat{A''B''}}{\widehat{A'B'}} \quad \text{or} \quad \varphi(x+y) = \varphi(x)\varphi(y).$$

The only solution of this equation is $\varphi(x) = e^{-\frac{x}{h}}$.
i.e. $\log \varphi(x+y) = \log \varphi(x) + \log \varphi(y)$, so $\log \varphi(x) = x \log \varphi(1)$ for x rational. Since $\log \varphi(x)$ is decreasing, this holds for all x (continuity).
Decreasing means $\log \varphi(1) < 0$, say $\log \varphi(1) = -\frac{1}{h}$, $h > 0$.
Then $\log \varphi(x) = -\frac{x}{h}$. <u>QED</u>.

<u>Remarks</u> 1) h depends only on the choice of unit segment UV.
2) h is the <u>radius of curvature</u> of the NE plane.

<u>Problem</u> (TH) A construction for h.

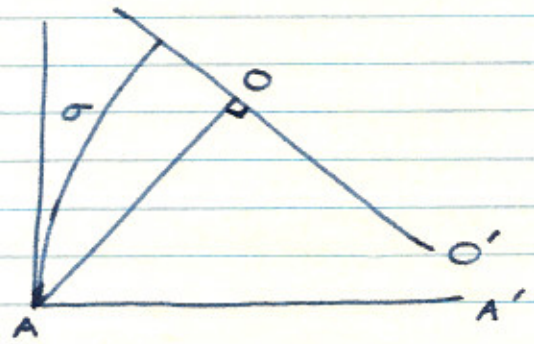


Let l be a radius of a horocycle through A.
Let $\widehat{AC}/\widehat{AB} = e$. Let the equidistant curve with l as axis and the radius through B is P.
If PQ and $CR \perp l$, then $QR = h$.
<u>Soln</u>: Scale l and construct horocycle through P.
$PQ = CR$ so $\widehat{SP} = \widehat{AC}$ and $\widehat{AB}/\widehat{SP} = \widehat{AB}/\widehat{AC} = e^{-1}$. This is $e^{-\frac{QR}{h}}$, so $QR = h$. $(QR = SA)$

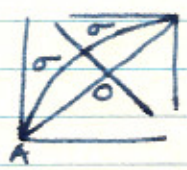
5/9 **Formulas for projection**

Preliminary remark. Given a point A on a horocycle, there is a unique radius of the horocycle parallel to the tangent to the horocycle at A.

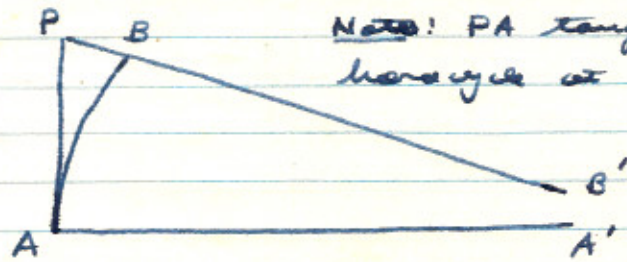
On a ray from A making angle $\frac{\rho}{2}$ with radius AA', choose O so $\frac{\rho}{2}$ is angle of parallelism for segment AO. OO' ∥ AA' is required radius. For O'O produced is parallel to the perpendicular to AA' at A.



Let σ denote the length of the arc cut off on the horocycle through A by radii AA' and OO'. σ is unique since 2σ is the arc subtended by a unique chord length 2AO. σ is another constant.
Later: We will see $\sigma = h$.



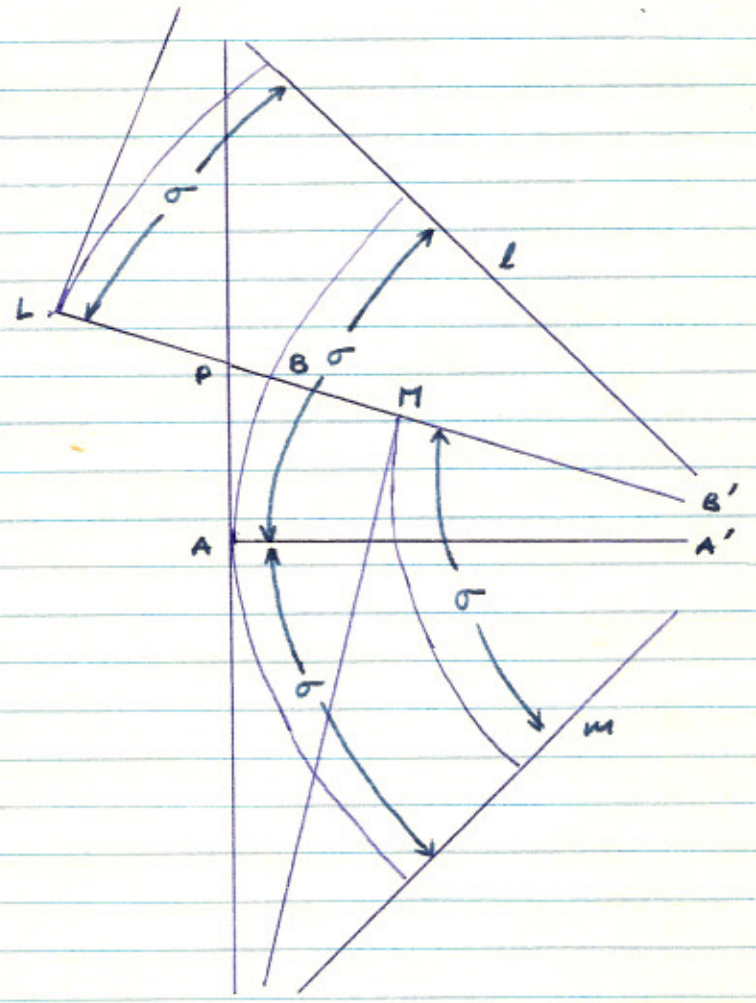
Formulas. Let radii AA' and BB' cut of arc $\widehat{AB} < \sigma$ on a horocycle. Let B'B produced meet the perpendicular to AA' at A in P. Then



Note: PA tangent to horocycle at A.

$$\widehat{AB} = \sigma \tanh \frac{AP}{h} \quad \text{and} \quad e^{\frac{BP}{h}} = \cosh \frac{AP}{h}.$$

Proof: Construction of the figure.
Since $\widehat{AB} < \sigma$, the tangent line to the horocycle arc at A meets B'B produced in P. PA ⊥ AA'.



i) Produce B'B from P to L so that PL = PA. The perpendicular to B'L at L is parallel to AP produced – by PL = PA and equal vertical angles of parallelism at P. Let l be parallel to both sides of the angle at L. Extend $\widehat{AB}$ to meet l and construct a concentric horocycle through L meeting l.

ii) Let M on PB' so that PM = PA. The perpendicular to PB' at M is parallel to PA produced – by PM = PA and coinciding angles of parallelism at P. Let m be parallel to both sides of the angle at M. Extend $\widehat{AB}$ to meet m and construct a concentric horocycle through M meeting m.

2) From the concentric horocycles meeting LB' and l:

$$\sigma - \widehat{AB} = \sigma e^{-\frac{LP+BP}{h}}.$$

From the concentric horocycles meeting LB' and m:

$$\sigma + \widehat{AB} = \sigma e^{\frac{MP-BP}{h}}.$$

From $AP = LP = MP$:

$$\sigma - \widehat{AB} = \sigma e^{-\frac{AP+BP}{h}}$$

$$\sigma + \widehat{AB} = \sigma e^{\frac{AP-BP}{h}}.$$

Add:

$$2\sigma = \sigma e^{-\frac{BP}{h}}\left(e^{\frac{AP}{h}} + e^{-\frac{AP}{h}}\right) = \sigma e^{-\frac{BP}{h}} 2\cosh\frac{AP}{h}$$

or

$$e^{\frac{BP}{h}} = \cosh\frac{AP}{h}.$$

Subtract first from second:

$$2\widehat{AB} = \sigma e^{-\frac{BP}{h}}\left(e^{\frac{AP}{h}} - e^{-\frac{AP}{h}}\right) = \sigma e^{-\frac{BP}{h}} 2\sinh\frac{AP}{h}$$

or

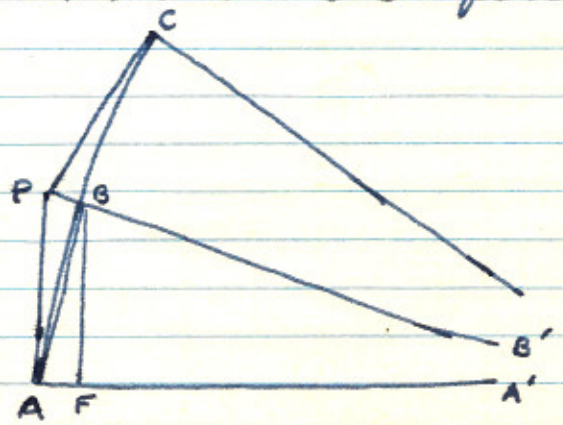
$$\widehat{AB} = \sigma\frac{\sinh\frac{AP}{h}}{e^{\frac{BP}{h}}} = \sigma\frac{\sinh\frac{AP}{h}}{\cosh\frac{AP}{h}} = \sigma\tanh\frac{AP}{h}.$$

QED.

Note Cf. Text §67 pp. 136–138.

Corollary The constants σ and h are equal: $\sigma = h$.

Proof. Construct on opposite side of PB' a congruent figure of the above: $AP = CP$, $\widehat{AB} = \widehat{CB}$. From the proof that horocycle arcs have finite length: $\widehat{AC} < AP + PC$ so $\widehat{AB} < AP$.



Let $BF \perp AA'$ at F. Then $\widehat{AB} > AB > FB$ so $FB < \widehat{AB} < AP$ and $\frac{FB}{AP} < \frac{\widehat{AB}}{AP} < 1$.

As AP tends to 0, $\lim \frac{FB}{AP} = 1$, so $\lim \frac{\widehat{AB}}{AP} = 1$.

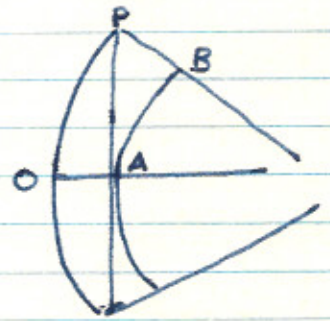
In fact from $e^{\frac{BP}{h}} = \cosh \frac{AP}{h}$, BP and FA also tend to 0. Now

$$\widehat{AB} = \sigma \tanh \frac{AP}{h} \quad \text{gives} \quad \frac{\widehat{AB}}{AP} = \frac{\sigma}{h} \frac{\tanh \frac{AP}{h}}{\frac{AP}{h}}.$$

Letting AP go to 0, we obtain $1 = \sigma/h$. QED

Problem (15) Text. p. 140 #5. If s is the length of the arc of a horocycle subtended by a chord of length l, show that $s = 2\sigma \sinh \frac{l}{2h}$.

Solution



$\widehat{OP} = \widehat{AB}\, e^{\frac{BP}{h}}$

$\widehat{AB} = \sigma \tanh \frac{AP}{h}$

From $e^{\frac{BP}{h}} = \cosh \frac{AP}{h}$,

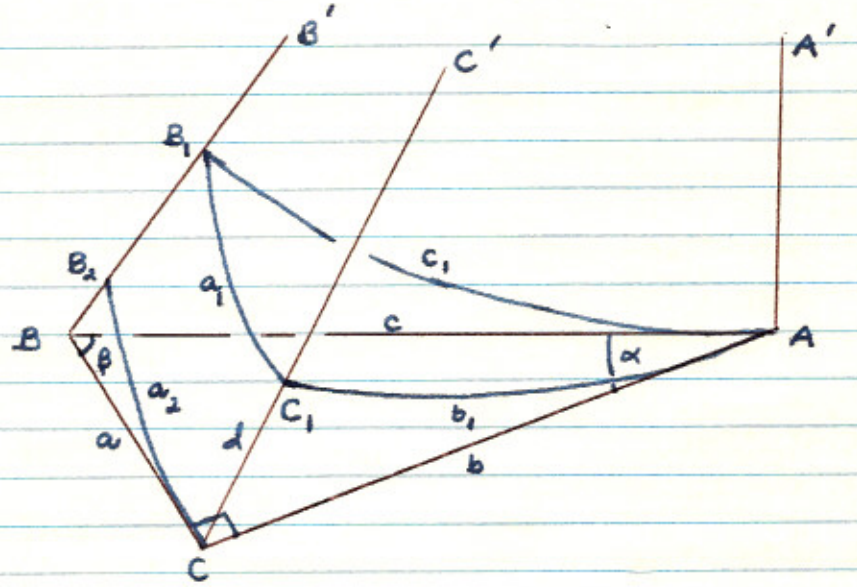
$\widehat{OP} = \sigma \tanh \frac{AP}{h} \cosh \frac{AP}{h}$

$= \sigma \sinh \frac{AP}{h}$.

Now use $\widehat{OP} = \frac{s}{2}$, $AP = \frac{l}{2}$.

5/12 Right triangles

Let ΔABC be a plane right triangle with: acute angles $\alpha = \angle A$, $\beta = \angle B$ and right angle $\rho = \angle C$, legs $a = BC$, $b = AC$, and hypotenuse $c = AB$. Let $AA' \perp$ plane ABC at A and let BB' and $CC' \parallel AA'$. These parallel lines determine a horosphere tangent to plane ABC at A. Plane $AA'BB'$ intersects the horosphere in $\widehat{AB_1}$ between AA' and BB'. Likewise planes $AA'CC'$ and $BB'CC'$. Obtain triangle ΔAB_1C_1 on the horosphere. ~~Prove right angle $\angle AC_1B_1 = \rho_1$ since plane $AA'CC' \perp$ plane ABC.~~ Now $AA' \perp$ plane ABC, so



plane $AA'CC' \perp$ plane ABC. BC is perpendicular to their line of intersection CA so $BC \perp$ plane $AA'CC'$. Thus plane $BB'CC' \perp$ plane $AA'CC'$ and dihedral angle CC' is a right angle. ΔAB_1C_1 is a right triangle with: $\angle A = \alpha$, $\angle C_1 = \beta$. Let $a_1 = \widehat{B_1C_1}$, $b_1 = \widehat{AC_1}$ be its legs, $c_1 = \widehat{AB_1}$ its hypotenuse. ΔAB_1C_1 is the projection of ΔABC onto the horosphere.

Let $\widehat{B_2C}$ be a horocycle arc concentric with $\widehat{B_1C_1}$ in plane $BB'CC'$; $a_2 = \widehat{B_2C}$ is its length, and $d = CC_1 = B_2B_1$ the distance between the horocycles.

From the first of the projection formulas:

$$b_1 = \sigma \tanh \frac{b}{h} \quad , \quad c_1 = \sigma \tanh \frac{c}{h}$$

and $a_2 = \sigma \tanh \frac{a}{h}$.

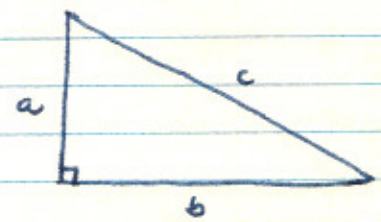
From the second of the projection formulas we get $e^{\frac{d}{h}} = \cosh \frac{b}{h}$. From $a_1 = a_2 e^{-\frac{d}{h}}$ we obtain

$$a_1 = \sigma \frac{\tanh \frac{a}{h}}{\cosh \frac{b}{h}} .$$

These formulas for a_1, b_1, c_1 allow one to obtain NE trigonometry formulas from Euclidean trigonometry.

<u>Pythagorean theorem.</u>

$$\cosh\frac{c}{h} = \cosh\frac{a}{h}\cosh\frac{b}{h}$$



<u>Proof</u> 1) Note from $\cosh^2 x - \sinh^2 x = 1$, obtain

$$1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \quad \cosh^2 x = \frac{1}{1-\tanh^2 x}.$$

2) From Euclidean Pythagorean theorem on the horosphere: $c_1^2 = a_1^2 + b_1^2$. So:

$$\tanh^2\frac{c}{h} = \frac{\tanh^2\frac{a}{h}}{\cosh^2\frac{b}{h}} + \tanh^2\frac{b}{h} \qquad \left(\begin{array}{c}\text{after division}\\ \text{by } \sigma^2\end{array}\right).$$

$$= \tanh^2\frac{a}{h}\left(1-\tanh^2\frac{b}{h}\right) + \tanh^2\frac{b}{h}$$

$$= \tanh^2\frac{a}{h} + \tanh^2\frac{b}{h} - \tanh^2\frac{a}{h}\tanh^2\frac{b}{h}$$

$$= 1 - \left(1-\tanh^2\frac{a}{h}\right)\left(1-\tanh^2\frac{b}{h}\right)$$

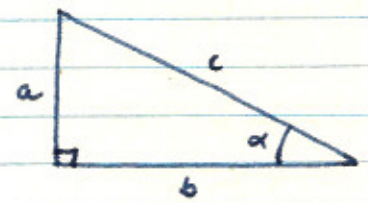
or

$$1 - \tanh^2\frac{c}{h} = \left(1-\tanh^2\frac{a}{h}\right)\left(1-\tanh^2\frac{b}{h}\right).$$

Taking reciprocals:

$$\cosh^2\frac{c}{h} = \cosh^2\frac{a}{h}\cosh^2\frac{b}{h}. \qquad \underline{\text{QED}}.$$

<u>Formulas relating two sides and an angle.</u>



$$\tanh\frac{b}{h} = \tanh\frac{c}{h}\cos\alpha$$

$$\sinh\frac{a}{h} = \sinh\frac{c}{h}\sin\alpha$$

$$\tanh\frac{a}{h} = \sinh\frac{b}{h}\tan\alpha$$

<u>Proof</u> 1) $b_1 = c_1\cos\alpha$ gives

$$\sigma\tanh\frac{b}{h} = \sigma\tanh\frac{c}{h}\cos\alpha$$

and the first formula.

2) $a_1 = c_1\sin\alpha$ gives

$$\sigma\frac{\tanh\frac{a}{h}}{\cosh\frac{b}{h}} = \sigma\tanh\frac{c}{h}\sin\alpha$$

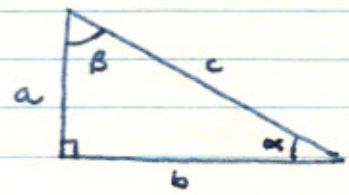
~~[illegible] formulas~~

so:
$$\frac{\sinh\frac{a}{h}}{\cosh\frac{a}{h}\cosh\frac{b}{h}} = \frac{\sinh\frac{c}{h}}{\cosh\frac{c}{h}}\sin\alpha .$$

The denominators are equal by the Pythagorean Theorem, so obtain second formula.

3) $a_1 = b_1 \tan\alpha$ gives
$$\sigma\,\frac{\tanh\frac{a}{h}}{\cosh\frac{b}{h}} = \sigma\tanh\frac{b}{h}\tan\alpha$$
and the third formula. QED.

<u>Formulas relating two angles and a side.</u>



$$\cos\alpha = \cosh\frac{a}{h}\sin\beta$$
$$\cosh\frac{c}{h} = \cot\alpha\cot\beta$$

<u>Remark</u> There are analogous formulas in spherical trigonometry but <u>not</u> Euclidean trigonometry since $\alpha+\beta=\rho$: $\cos\alpha = \sin\beta$, $\cot\alpha\cot\beta = 1$.

<u>Proof.</u> 1)
$$\tanh\frac{a}{h} = \sinh\frac{b}{h}\tan\alpha$$
$$\tanh\frac{b}{h} = \sinh\frac{a}{h}\tan\beta$$
Multiply:
$$\frac{\sinh\frac{a}{h}}{\cosh\frac{a}{h}}\,\frac{\sinh\frac{b}{h}}{\cosh\frac{b}{h}} = \sinh\frac{b}{h}\sinh\frac{a}{h}\tan\alpha\tan\beta .$$
Pythagorean Theorem gives $\dfrac{1}{\cosh\frac{c}{h}} = \tan\alpha\tan\beta$

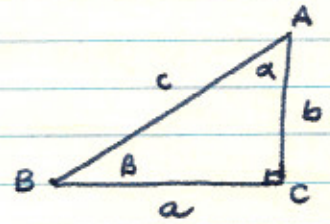
and the second formula.

2)
$$\tanh\frac{b}{h} = \tanh\frac{c}{h}\cos\alpha$$
$$\sinh\frac{b}{h} = \sinh\frac{c}{h}\sin\beta \quad \left(\begin{array}{l}\text{second formula}\\ \text{above for } \beta\end{array}\right)$$
Divide:
$$\frac{1}{\cosh\frac{b}{h}} = \frac{1}{\cosh\frac{c}{h}}\,\frac{\cos\alpha}{\sin\beta}$$
Pythagorean theorem gives: $\dfrac{\cosh\frac{a}{h}}{\cosh\frac{c}{h}} = \dfrac{1}{\cosh\frac{c}{h}}\,\dfrac{\cos\alpha}{\sin\beta}$

and the first formula. QED.

Remark. These formulas may be summarized by Napier's rules as in spherical trigonometry. One uses hyperbolic functions rather than circular functions when for lengths.

Summary of formulas



$$\cosh\frac{c}{h} = \cosh\frac{a}{h}\cosh\frac{b}{h}$$

$$\cosh\frac{c}{h} = \cot\alpha\cot\beta$$

$$\tanh\frac{a}{h} = \tanh\frac{c}{h}\cos\beta \qquad \tanh\frac{b}{h} = \tanh\frac{c}{h}\cos\alpha$$

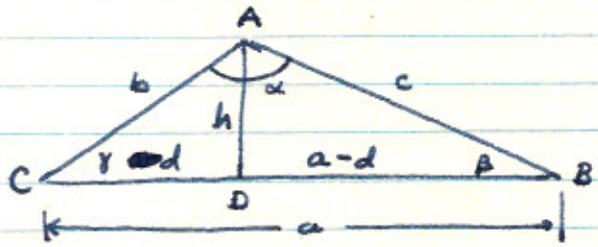
$$\sinh\frac{b}{h} = \sinh\frac{c}{h}\sin\beta \qquad \sinh\frac{a}{h} = \sinh\frac{c}{h}\sin\alpha$$

$$\tanh\frac{b}{h} = \sinh\frac{a}{h}\tan\beta \qquad \tanh\frac{a}{h} = \sinh\frac{b}{h}\tan\alpha$$

$$\cos\beta = \cosh\frac{b}{h}\sin\alpha \qquad \cos\alpha = \cosh\frac{a}{h}\sin\beta$$

Cf Text §73 p. 152.

General triangles



Let $AD \perp BC$ at D
(figure drawn
with D between B & C).
$h = AD$, altitude.

Law of Cosines. $$\cosh\frac{c}{h} = \cosh\frac{a}{h}\cosh\frac{b}{h} - \sinh\frac{a}{h}\sinh\frac{b}{h}\cos\gamma.$$

Proof Note: $\cosh(x-y) = \cosh x\cosh y - \sinh x\sinh y$
Then:

$$\cosh\frac{c}{h} = \cosh\frac{h}{h}\cosh\frac{a-d}{h} = \cosh\frac{h}{h}\left(\cosh\frac{a}{h}\cosh\frac{d}{h} - \sinh\frac{a}{h}\sinh\frac{d}{h}\right)$$

$$= \cosh\frac{a}{h}\underbrace{\cosh\frac{h}{h}\cosh\frac{d}{h}}_{\cosh\frac{b}{h}}\left(1 - \tanh\frac{a}{h}\underbrace{\tanh\frac{d}{h}}_{\tanh\frac{b}{h}\cos\gamma}\right)$$

$$= \cosh\frac{a}{h}\cosh\frac{b}{h}\left(1 - \tanh\frac{a}{h}\tanh\frac{b}{h}\cos\gamma\right)$$

$$= \cosh\frac{a}{h}\cosh\frac{b}{h} - \sinh\frac{a}{h}\sinh\frac{b}{h}\cos\gamma. \qquad \text{QED.}$$

5/14 <u>Law of sines.</u> $\dfrac{\sin\alpha}{\sinh\frac{a}{h}} = \dfrac{\sin\beta}{\sinh\frac{b}{h}} = \dfrac{\sin\gamma}{\sinh\frac{c}{h}}$.

<u>Proof</u>

$$\sinh\frac{b}{h} = \sinh\frac{b}{h}\sin\gamma \quad = \sinh\frac{c}{h}\sin\beta$$

(triangle on left) (triangle on right)

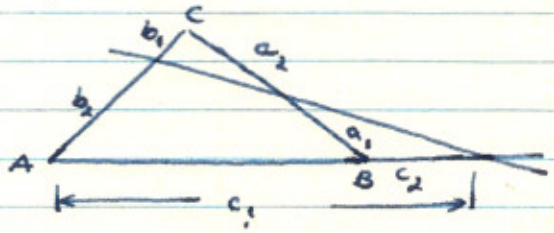
so $\sin\beta / \sinh\frac{b}{h} = \sin\gamma / \sinh\frac{c}{h}$. The other similarly. QED

(over) →

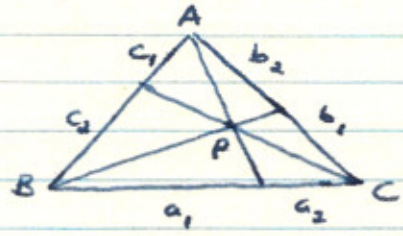
<u>Problem</u> (16). Look up the proofs of the Theorems of Menelaus and Ceva in Euclidean geometry, then work Exercises Text p. 147-148 #1, #3.

<u>Ref.</u> N. A. Court. College Geometry, Barnes & Noble, New York, 1952. Paras. 309 ff, 325 ff.

<u>p. 147 #1.</u> If a transversal cuts the sides of triangle ABC, dividing side a into segments a_1 and a_2, side b into segments b_1 and b_2, and side c into c_1 and c_2, prove that $\sinh\frac{a_1}{h}\sinh\frac{b_1}{h}\sinh\frac{c_1}{h} = \sinh\frac{a_2}{h}\sinh\frac{b_2}{h}\sinh\frac{c_2}{h}$. (The converse of the Theorem of Menelaus.)



<u>p. 147 #3</u> If the lines joining point P to the vertices of triangle ABC divide the sides into pairs of segments a_1 and a_2, b_1 and b_2, c_1 and c_2, prove that



$$\sinh\frac{a_1}{h}\sinh\frac{b_1}{h}\sinh\frac{c_1}{h} = \sinh\frac{a_2}{h}\sinh\frac{b_2}{h}\sinh\frac{c_2}{h}$$

(Compare with the Theorem of Ceva.)

<u>Remark</u> "Law of cosines" for angles:

$$\cosh \frac{c}{k} = \frac{\cos\alpha \cos\beta + \cos\gamma}{\sin\alpha \sin\beta}, \text{ etc.}$$

<u>See</u>: H. Busemann, Hyperbolic Geometry, Academic Press, New York, 1964, p. 85.

Relationship to Euclidean and spherical trigonometry

Euclidean. Using Maclaurin series

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

the Law of Cosines

$$\cosh \frac{c}{h} = \cosh \frac{a}{h} \cosh \frac{b}{h} - \sinh \frac{a}{h} \sinh \frac{b}{h} \cos\gamma$$

becomes

$$1 + \frac{c^2}{2h^2} + \cdots = \left(1 + \frac{a^2}{2h^2} + \cdots\right)\left(1 + \frac{b^2}{2h^2} + \cdots\right)$$

$$- \left(\frac{a}{h} + \cdots\right)\left(\frac{b}{h} + \cdots\right) \cos\gamma$$

where terms with $1/h^3$ and higher order are omitted. Cancel 1 and multiply by $2h^2$:

$$c^2 + \cdots = (a^2 + b^2 + \cdots) - (2ab + \cdots) \cos\gamma$$

where terms with $1/h$ and higher order are omitted. Let $h \to \infty$:

$$c^2 = a^2 + b^2 - 2ab \cos\gamma.$$

This is the Euclidean Law of Cosines.

Similarly: Other formulas yield Euclidean versions as $h \to \infty$.

Interpretation: 1) As h becomes infinitely large, NE geometry becomes Euclidean geometry.

2) NE geometry is Euclidean in the small: When lengths are small compared with h, Euclidean trigonometry formulas almost hold.

Spherical. Using the relations

$$\sin i x = i \sinh x, \quad \cos i x = \cosh x$$

the spherical Law of Cosines for sphere of radius R

$$\cos \frac{c}{R} = \cos \frac{a}{R} \cos \frac{b}{R} + \sin \frac{a}{R} \sin \frac{b}{R} \cos\gamma$$

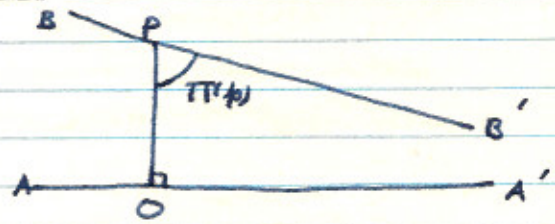
becomes, for $R = \sqrt{-1}\,k$:

$$\cosh\frac{c}{k} = \cosh\frac{a}{k}\cosh\frac{b}{k} - \sinh\frac{a}{k}\sinh\frac{b}{k}\cos\gamma.$$

<u>Similarly</u>: Other spherical formulas yield NE version for $R = \sqrt{-1}\,k$.

<u>Interpretation</u>: NE geometry is spherical geometry on a sphere of imaginary radius $R = \sqrt{-1}\,k$.

<u>Angle of parallelism formula</u>
(Lobachevsky - Bolyai)



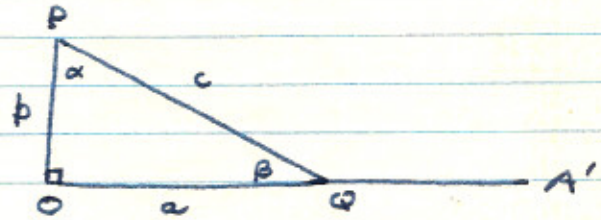
$p = PO$

$\Pi(p) = \angle B'PO$

$$\tan\frac{\Pi(p)}{2} = e^{-\frac{p}{k}}$$

<u>Proof</u> 1) Let Q on OA', $a = OQ$.
Note that

$\alpha \to \Pi(p)$ as $OQ \to \infty$.



2) From

$$\sin\beta = \frac{\cos\alpha}{\cosh\frac{a}{k}} < \frac{1}{\cosh\frac{a}{k}}$$

conclude: $\beta \to 0$ as $OQ \to \infty$.
From

$$\sin\alpha = \frac{\cos\beta}{\cosh\frac{p}{k}}, \quad \cot\alpha = \frac{\tanh\frac{p}{k}}{\tanh\frac{c}{k}}$$

conclude as $OQ \to \infty$:

$$\sin\Pi(p) = \frac{1}{\cosh\frac{p}{k}}, \quad \cos\Pi(p) = \tanh\frac{p}{k}$$

3) From $\tan\frac{x}{2} = \frac{1-\cos x}{\sin x}$

we get

$$\tan\frac{\Pi(p)}{2} = \frac{1-\frac{\sinh\frac{p}{h}}{\cosh\frac{p}{h}}}{\frac{1}{\cosh\frac{p}{h}}} = \cosh\frac{p}{h} - \sinh\frac{p}{h} = e^{-\frac{p}{h}}.$$

Note: As $h \to \infty$, $\tan\frac{\Pi(p)}{2} = 1$ gives $\frac{\Pi(p)}{2} = \frac{\pi}{4}$ (mod ω)
or $\Pi(p) = \frac{\pi}{2}$ (mod ω) $= \rho$ for Euclidean geometry.

Remark. Lobachevsky worked with the angles of parallelism and circular functions. His form of the Law of Cosines was:

$$\sin\Pi(c) = \frac{\sin\Pi(a)\sin\Pi(b)}{1-\cos\Pi(a)\cos\Pi(b)\cos\gamma}.$$

Problem (17) a) Show $\Pi(p) = \frac{\rho}{2} = \frac{\pi}{4}$ (mod ω) $= 45°$ for $\frac{p}{h} = \log(\sqrt{2}+1)$.
b) Derive Lobachevsky's form of Law of Cosines.
Solution: a) $\tan\frac{\pi}{8} = \frac{1-\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}} = \frac{1-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \sqrt{2}-1$. (over) →

$\sqrt{2}-1 = e^{-\frac{p}{h}}$ yields $e^{\frac{p}{h}} = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1$.
b) Use

$$\sin\Pi(p) = \frac{1}{\cosh\frac{p}{h}}, \quad \cos\Pi(p) = \tanh\frac{p}{h};$$

which yield

$$\sinh\frac{p}{h} = \frac{\cos\Pi(p)}{\sin\Pi(p)}, \quad \cosh\frac{p}{h} = \frac{1}{\sin\Pi(p)}.$$

$$\cosh\frac{c}{h} = \cosh\frac{a}{h}\cosh\frac{b}{h} - \sinh\frac{a}{h}\sinh\frac{b}{h}\cos\gamma$$

becomes:

$$\frac{1}{\sin\Pi(c)} = \frac{1}{\sin\Pi(a)}\frac{1}{\sin\Pi(b)} - \frac{\cos\Pi(a)}{\sin\Pi(a)}\frac{\cos\Pi(b)}{\sin\Pi(b)}\cos\gamma$$

which immediately gives Lobachevsky's formula.
c) Sine formula is special case of b). Cosine formula follows from $\tanh\frac{b}{h} = \tanh\frac{c}{h}\cos\alpha$ and $\cos\Pi(b) = \tanh\frac{b}{h}$, etc.

(over) →

~~Remark "Law of cosines" for angles:~~

$$\cosh \frac{c}{k} = \frac{\cos\alpha \cos\beta + \cos\gamma}{\sin\alpha \sin\beta}, \text{ etc.}$$

~~See H. Busemann, Non-euclidean geometry, Academic Press, New York, 1964, p. 85.~~

b) Prove, for a right triangle with c the hypotenuse:

sine formula : $\sin \Pi(c) = \sin \Pi(a) \sin \Pi(b)$

cosine formula : $\cos \Pi(b) = \cos \Pi(c) \cos \alpha$

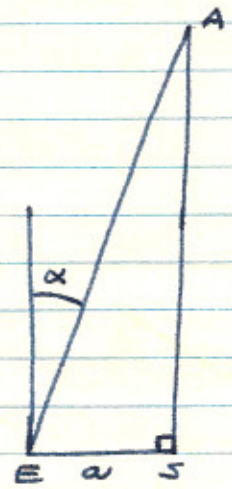
Remark

$$\sin \Pi(x \pm y) = \frac{\sin \Pi(x) \sin \Pi(y)}{1 \pm \cos \Pi(x) \cos \Pi(y)}$$

$$\cos \Pi(x \pm y) = \frac{\cos \Pi(x) \cos \Pi(y)}{1 \pm \cos \Pi(x) \cos \Pi(y)}, \text{ etc.}$$

See Borsuk - Szmielew, Foundations of Geometry, Ch. VI, §26, pp. 334 ff.

Astronomical estimation of k. (Lobachevsky).



S = Sun
E = Earth
A = star
a = SE, radius of Earth's orbit
α = parallax of star, angle between perpendicular to ES at E and EA.

Claim: $k > \dfrac{a}{\tan\alpha}$

Proof 1) An inequality: For $0 < x < 1$:

$$\log\frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right)$$

$$\frac{2x}{1-x^2} = 2\left(x + x^3 + x^5 + x^7 + \cdots\right)$$

so

$$\log\frac{1+x}{1-x} < \frac{2x}{1-x^2} \quad \text{for } 0 < x < 1.$$

Assume: Light moves along straight lines.

2) Since EA meets SA at A,

$$\Pi(a) > \angle AES = \frac{\pi}{2} - \alpha, \text{ and } \frac{\Pi(a)}{2} > \frac{\pi}{4} - \frac{\alpha}{2}.$$

Then

$$e^{-\frac{a}{k}} = \tan\frac{\Pi(a)}{2} > \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) = \frac{\tan\frac{\pi}{4} - \tan\frac{\alpha}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{\alpha}{2}} = \frac{1 - \tan\frac{\alpha}{2}}{1 + \tan\frac{\alpha}{2}}$$

and

$$e^{\frac{a}{k}} < \frac{1 + \tan\frac{\alpha}{2}}{1 - \tan\frac{\alpha}{2}}.$$

Using the inequality in 1):

$$\frac{a}{k} < \log\frac{1 + \tan\frac{\alpha}{2}}{1 - \tan\frac{\alpha}{2}} < \frac{2\tan\frac{\alpha}{2}}{1 - \tan^2\frac{\alpha}{2}} = \tan\alpha.$$

QED.

5/16 Numerical calculation

Using star Sirius (approx. 8.7 light years from Earth) with parallax $\alpha = 1.24''$ (Lobachevsky's data) we have:

$$\tan\alpha \approx \alpha^{(rad)} = \frac{1.24}{3600} \times \frac{\pi}{180} = 6.012 \times 10^{-6}$$

$$a = 1.496 \times 10^{13} \text{ cm} = 92\,960\,000 \text{ miles}$$

$$\frac{a}{\tan\alpha} = \frac{1.496 \times 10^{13}}{.6012 \times 10^{-5}} = 2.488 \times 10^{18} \text{ cm} = 1.546 \times 10^{13} \text{ miles}$$

Now:

1 light-year $= 0.94605 \times 10^{18}$ cm $= 5.879 \times 10^{12}$ miles

so

$$\frac{a}{\tan\alpha} = \frac{2.488 \times 10^{18}}{.94605 \times 10^{18}} = \frac{15.46 \times 10^{12}}{5.879 \times 10^{12}} = 2.630 \text{ lt-yrs.}$$

<u>Thus</u>: $h > 2.63$ light-years.

<u>Remark</u> Easy to calculate: $h > \dfrac{3.262}{\alpha^{(sec)}}$ light-years

If we find a star with parallax $\alpha = 0.1''$ say, then we can conclude ~~h > 3260 light-years~~.
$h > 2\,000\,000\,a$ <u>i.e.</u> $h > 32.6$ light-years.

<u>Note</u>: For d = Euclidean distance to star A, $d = \dfrac{a}{\tan\alpha}$.
so $h > d$.

<u>From</u>: Norton's Star Atlas.

	α	d
Sirius	0".371	9 light-years
β Centauri	0".011	300
Aratures	0".009	360

For Sirius: $3.262 / .371 = 8.79$ light-years.

<u>Note</u>: 1" of arc is subtended by a at 1 parsec.
1 parsec = 3.262 light-years.