7. *Exercises on curves and consistency*

Ref. Text: Ch. VIII.

Note: Text refers in terms of the Poincaré model.

**Projections onto a hyperplane**

In NE space, let a hyperplane be defined by point O and the bundle of lines passing to 00' let x be plane tangent to the hyperplane x 0 0' 1 x.

For P and  x, let 00' must be a hyperplane at 0. Then obtain nothing:

P 0 0', plane x to hyperplane.

**In the projection of x onto the hyperplane**.

Note: Only points P, of hyperplane for which 00' < 0 are obtained.

Thus! Image is interior of circle x

Then (circle) of circle of 0.

Clearly: Lines in x

Project. To hyperplane are

Lines in the circle.

We see the Euclidean

hyperplane:

- circle = circle of interior x
- points = points interior to the circle
- line = chord of the circle (less endpoints).
Dispersion of lines

1) Intersecting lines

Claim: If lines m and n intersect in P in \( \pi \), their images \( l_1 \) and \( m_1 \) intersect in the image \( P_1 \) of P. Conversely.

2) Parallel lines

Claim: If lines m and n are parallel in \( \pi \), their images \( l_1 \) and \( m_1 \) intersect in a point \( Q_1 \) on the absolute. Conversely.

Proof: If \( \pi \), let \( AA' \) be a given line. Let \( CC' \) be a line through \( O \), \( CC' \parallel AA' \).

Let \( QQ' \parallel CC' \), and \( Q'Q \parallel CC' \); \( QQ' \) meets the absolute in \( Q_1 \), and clearly \( CQ_1 = r \), so \( Q \),

is on the absolute. Plane \( CC'QQ' \) meets the absolute in the image of CC'. Now: AA' \parallel CC' and \( Q'Q \parallel CC' \).

\( AA' \parallel OQ' \) and CC' \parallel QQ' lies in a plane.

Plane \( AA'QQ' \) meets the absolute in image of AA'.

Since plane \( AA'QQ' \) and CC' QQ' meet in \( Q' \), \( Q' \) is the image of AA' and CC' meet in \( Q_1 \).

If \( AA' \parallel BB' \) in \( \pi \), let CC' \parallel BB' to pass through \( O_1 \) on the image of \( AA' \) and \( BB' \) lies in the plane Q, Q' above.

The image of \( BB' \) lies in the plane Q, Q' above.
3) Non-intersecting lines.

Claim: If \( l \) and \( m \) are non-intersecting lines

\[ \alpha \cup l \cup m \]

and \( m \) intersect at a point outside the planes \( l \) and \( m \), we have that the

plane \( l \) and \( m \) are parallel in the Euclidean sense. If \( l \) and \( m \) intersect on the line \( o \), then \( l \) and \( m \) are parallel in the projection plane. Conversely.

Proof:

Let \( AA' \) and \( BB' \) be non-intersecting, \( MN \) their
common perpendicular. \( M \) on \( AA' \), \( N \) on \( BB' \).
Let \( CC' \) through \( O \), \( CC' \perp MN \) at \( R \). Through
\( MN \) let plane \( \beta \) through \( O \).
Let \( QQ' \) parallel to \( OO' \) and perpendicular to \( \beta \\
meet \( \beta \) at \( S \). Such a line \( QQ' \) exists if \n\( RO \) exists. If \( MN \) does
not pass through \( O \).
Name: \( AA' \) and \( QQ' \) are
the same perpendicular to \( \beta \), so \( AA' \) is a line.
Plane \( AA'QQ' \) meets \( MN \) at a point in the line \( O \).
Since plane \( CC'QQ' \) meets \( MN \) at a point in the line \( O \),
\( MN \cup AA' \cap CC' \cap QQ' \) meet \( MN \) at a point in the line \( O \).
\( MN \cup AA' \cap CC' \cap QQ' \) meet \( MN \) at a point in the line \( O \).
So \( MN \cup AA' \cap CC' \cap QQ' \) meet \( MN \) at a point in the line \( O \).
By symmetry, \( MN \cup BB' \cap CC' \cap QQ' \) meet \( MN \) at a point in the line \( O \).
2) Through the method of RS
there is a line parallel to QQ' and CC'. It
is then evident that QQ' > 0.

3) If R = 0, then the planes of the conical through
AA' and BB' meet at in parallel lines, so these
planes are parallel. They are not coincident, so
the images of AA' and BB' do not meet. Q.E.D.

Remark: Conics may be found directly using the law of
traces and the fact that these coincide maximally.

The distance formula

Notation as before. \( P_i \) denotes the image of \( P \)
with projection to the sphere tangent at \( C \).

1) Let \( P \) and \( Q \)
be two points on \( \alpha \), \( d = PQ \)
the distance
between them, and \( \theta = \angle P Q O \).

Let \( P_i \) and \( Q_i \) be
their images on
the horizon plane.

Then:\( \theta = \angle P_i O_i Q_i \).

Then:\[
\cos \frac{d}{n} = \frac{r^2 - OP_i O_i \cos \theta}{\sqrt{r^2 - OP_i^2} \sqrt{r^2 - O_i Q_i^2}}
\]

Proof. From law of cosines for \( \Delta POQ \):
\[
\cos \frac{d}{n} = \cos \frac{OP}{n} \cos \frac{OQ}{n} - \sin \frac{OP}{n} \sin \frac{OQ}{n} \cos \theta.
\]

From the first projection formula:
\[ \cos \theta_1 = \sigma \tan \theta_2 \]

\[ \cos \theta_2 = \sigma \tan \theta_1 \]

\[ \cos \theta_1 = \frac{1}{\sqrt{1 - \left(\frac{\sigma}{\sigma_0}\right)^2}} = \frac{\sigma}{\sqrt{\sigma^2 - \sigma_0^2}} \]

\[ \cos \theta_2 = \frac{\sigma_0}{\sqrt{1 - \left(\frac{\sigma}{\sigma_0}\right)^2}} = \frac{\sigma}{\sqrt{\sigma^2 - \sigma_0^2}} \]

and similarly for \( \cos \theta_2 \) and \( \sin \theta_2 \).

Laws of Cosines gives:

\[ \cos \theta_1 = \frac{\sigma_0}{\sqrt{\sigma^2 - \sigma_1^2}} \]  \[ \cos \theta_2 = \frac{\sigma}{\sqrt{\sigma^2 - \sigma_2^2}} \]

2) In the Euclidean plane, each is the hemisphere, introduce rectangular Cartesian coordinate \( x \) and \( y \) so that the abscissas in \( x^2 + y^2 = \sigma^2 \). Points that are the images of points of \( x \) satisfy \( x^2 + y^2 < \sigma^2 \).

Note: Unit of length is determined by the unit of length in \( x \) and \( y \) projections.

Let the images of \( P \) and \( Q \) is a plane coordinates

\[ P_1 = (x_1, y_1), \quad Q_1 = (x_2, y_2) \]

Then the NE distance \( d = PQ \) between \( P \) and \( Q \)

\[ \cos \theta_1 = \frac{\sigma^2 - x_1 x_2 - y_1 y_2}{\sigma^2 - x_1^2 - y_1^2 - \sqrt{\sigma^2 - x_2^2 - y_2^2}} \]

\[ \cos \theta_2 = \frac{\sigma_0}{\sqrt{\sigma^2 - \sigma_1^2}} \]  \[ \cos \theta_2 = \frac{\sigma}{\sqrt{\sigma^2 - \sigma_2^2}} \]
Proof: From Euclidean analytic geometry:
\[ O_1^2 = x_1^2 + y_1^2 \quad \text{and} \quad O_2^2 = x_2^2 + y_2^2. \]

Now use previous formulas. \( \square \)

Remark on cross ratios:

a) In the Euclidean plane, the cross ratio (harmonic ratio) of four collinear points \( P, N, A, B \) is defined by
\[ (P_1N_1A_1B_1) = \frac{PA}{AN} / \frac{MB}{BN}. \]

The distances are taken with signs.

b) If \( P_1, N_1, A_1, B_1 \) are parallel projections of \( P, N, A, B \), clearly
\[ (P_1N_1A_1B_1) = (P_1N_1A_1B_1). \]

If \( P_1, N_1, A_1, B_1 \) are perspective from \( V \), using \( P_1, N, A, B \), then
\[ (P_1N_1A_1B_1) = (P_1N_1A_1B_1). \]

This is proven by showing
\[ \frac{PA}{AN} / \frac{MB}{BN} = \frac{\sin \angle NVA}{\sin \angle NVB} / \frac{\sin \angle ANV}{\sin \angle BNV}; \]

so it depends only on angles at \( V \).

c) The iteration of harmonic ratios along between lines gives projective transformations between lines. These also come from projective transformations of the plane. The cross ratio \( (P_1N_1A_1B_1) \) is a projective invariant.

3) Let the line joining \( P, Q_1 \),
the midpoints of \( P, Q \) meet
the altitude in \( P, N \). Then
the NE distance \( d = PA \)

is given by
\[ d = \pm \frac{1}{2} \log (PQ, Q_1). \]
Proof:  
1. Let:  
   \[ A = \sigma^2 - x^2 - y^2 \]
   \[ B = \sigma^2 - x_1 x_2 - y_1 y_2 \]
   \[ C = \sigma^2 - x_2^2 - y_2^2 \]

   Then formula in 2) reads:  
   \[ \cosh \frac{\theta}{2} = \frac{B}{\sqrt{AC}} \]

2. Line joining \( P_1 = (x_1, y_1) \) and \( P_2 = (x_2, y_2) \):  
   \[ x = (1-t)x_1 + tx_2 \]
   \[ y = (1-t)y_1 + ty_2 \]
   \[ t = 0 \Rightarrow P_1, \quad t = 1 \Rightarrow P_2. \]

   Line meets the absolute circle  
   \[ x^2 + y^2 = \sigma^2 \]
   \[ (1-t)^2(x_1^2 + y_1^2) + 2(1-t)x_1(x_2 + y_2) + t^2(x_2^2 + y_2^2) = \sigma^2. \]

   Subtract this from  
   \[ (1-t)^2 \sigma^2 + \sigma^2 - x_1^2 \sigma^2 - x_2^2 \sigma^2 = \sigma^2 \]

   to obtain:  
   \[ (1-t)^2 A + 2(1-t)x_1 B + t^2 C = 0. \]

3. The roots of \( \# \) are:  
   \[ t = u < 0 \] for \( M \)
   \[ t = u > 1 \] for \( N \).

   The cross ratio is  
   \[ \lambda = (MNQO) = \frac{MP_1}{PN} / \frac{PQ}{QO} = \frac{u}{0} = \frac{1-w}{w} = \left( \frac{1-\frac{1}{w}}{\frac{1}{w}} \right). \]

   Note:  
   \[ \frac{1-w}{w} < \frac{1-\frac{1}{w}}{\frac{1}{w}} < 0 \quad \text{and} \quad 0 < w < 1. \]

   From \( \# \)  
   \[ A \left( \frac{1-t}{t} \right)^2 + B \frac{1-t}{t} + C = 0 \]

   and \( \lambda \) is the ratio of the roots of this equation.

4. Let  
   \[ u = \frac{1-w}{w} \]

   be one root.  

   Then  
   \[ u + \lambda u = -\frac{B}{A} \]

   sum of roots  
   \[ \lambda u^2 = \frac{C}{A} \]

   product of roots.

   From equation given  
   \[ (\lambda + 1)u^2 = \frac{4B^2}{A^2} \]

   Divide this by second equation to obtain.
\[
\frac{(\lambda+1)^2}{\alpha} = \frac{\mu \theta^2}{\alpha c} \quad \text{or} \quad \lambda + 1 + \lambda^{-1} = \frac{\mu \theta^2}{\alpha c}.
\]

This is
\[
\left(\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}\right)^2 = \frac{\mu \theta^2}{\alpha c} \quad \text{or} \quad \frac{1}{2} \left(\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}\right) = \frac{B}{\alpha c}.
\]

e) Comparing \[ \frac{d}{dh} = \frac{1}{2} \left( e^{\frac{1}{h}} + e^{-\frac{1}{h}} \right) = \frac{B}{\alpha c} \quad \text{and} \quad \frac{1}{2} \left( \lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}\right) = \frac{B}{\alpha c}. \]

To conclude
\[
e^{-\frac{1}{h}} = \lambda^{\frac{1}{2}} \quad \text{or} \quad \lambda^{-\frac{1}{2}} \quad \Rightarrow \quad \frac{d}{dh} = \pm \frac{1}{2} \log \lambda. \quad \text{QED.}
\]

The angle formula.

Notations as before. \( P \), denotes the moving of a point on the area of the line \( \lambda \), \( A \), denotes the moving of \( \alpha \). \( l \), denotes the moving of a line \( l \), in \( \alpha \).

Let \( x \) and \( y \) be

Cartesian coordinates as before. Then

\[ ux + vy - w = 0 \]

is a line. If

\[ \frac{w}{\sqrt{u^2 + v^2}} < 0 \quad \text{or} \quad \sigma^2 (u^2 + v^2) - w^2 > 0 \]

it intersects the abscissa and is the moving \( E \), of a line \( l \), in \( \alpha \).

1) If \[ l_1: ux + vy - w_1 = 0 \]
\[ w_1 = \frac{u^2 + v^2}{w} \]

intersects in \( x^2 + y^2 < \sigma^2 \), they are the moving of intersecting lines \( l \) and \( w \) in \( \alpha \).

If \( \psi \) denotes the angle between \( l \) and \( w \) in \( \alpha \), then

\[
\cos \psi = \pm \frac{\sigma^2 (u_2 x + v_2 y) - w_1 w_2}{\sqrt{\sigma^2 (u_2^2 + v_2^2) - w_2^2} \sqrt{\sigma^2 (u_1^2 + v_1^2) - w_1^2}}.
\]
Problem 15: Prove the angle formulae above. Sketch - direct algebraic solution.

1) Let \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) meet at \( \mathbf{P} \). \( \mathbf{P} \) has coordinates

\[
\mathbf{x}_0 = \frac{u_1 v_2 - u_2 v_1}{u_1 v_2 - u_2 v_1}, \quad \mathbf{y}_0 = \frac{v_1 u_2 - v_2 u_1}{u_1 v_2 - u_2 v_1}.
\]

Let \( OL = L \); then \( \mathbf{O}L, \mathbf{L} \), in minds.

\[
\mathbf{x}_1 = \frac{u_1}{u_2^2 + v_2^2} u_2, \quad \mathbf{y}_1 = \frac{v_1}{u_2^2 + v_2^2} v_2, \quad \mathbf{N}_A = \frac{u_1}{u_2^2 + v_2^2}
\]

Similarly construct \( \mathbf{M}_{1} \). \( \mathbf{M}_{1} \) has coordinates

\[
\mathbf{x}_2 = \frac{u_2}{u_2^2 + v_2^2} u_2, \quad \mathbf{y}_2 = \frac{v_2}{u_2^2 + v_2^2} v_2.
\]

2) Calculate \( \theta \) from law of cosines:

\[
\cos \frac{\mathbf{L} \mathbf{M}}{h} = \cos \frac{\mathbf{L} \mathbf{O}}{h} \cos \frac{\mathbf{P} \mathbf{M}}{h} - \sin \frac{\mathbf{L} \mathbf{O}}{h} \sin \frac{\mathbf{P} \mathbf{M}}{h} \cos \theta.
\]

Set \( A_{ij} = h^2 - x_i x_j - y_i y_j \), so

\[
\cos \frac{\mathbf{L} \mathbf{M}}{h} = \frac{A_{12}}{\sqrt{A_{01} A_{44}}}, \text{ etc.}
\]

Then

\[
\frac{A_{12}}{\sqrt{A_{01} A_{44}}} = \frac{A_{01} - A_{02} A_{11}}{\sqrt{A_{01} A_{11} - A_{02} A_{12}}} \cos \theta = A_{01} A_{11} - A_{02} A_{12}.
\]

Long reduction then gives formulae. (Not checked.)

Sketch - direct algebraic solution.

1) Let \( \mathbf{O}L = L \), \( \mathbf{O}M = M \) at \( \mathbf{P} \). \( \mathbf{P} \) and \( \mathbf{O} \) are on line, \( \mathbf{O} \) on \( \mathbf{P} \).

From \( \beta \),

\[
\cos \beta + \theta = \cos \frac{OL}{h} \sin \frac{MP}{h}.
\]

and long reduction, obtain

\[
\pm \cos \theta = \cos \frac{OL}{h} \cos \frac{OM}{h} \cos \theta - \sin \frac{OL}{h} \sin \frac{OM}{h}.
\]

2) Use \( \mathbf{O}L = \frac{h}{\tan \theta}, \mathbf{OM} = \frac{h}{\tan \theta} \), on before to obtain...
\[
\cos \theta = \pm \frac{\cos \phi - \cos \psi}{\sqrt{\cos^2 \phi - \cos^2 \psi}}.
\]

3) Use
\[
\cos \theta = \frac{u_1 u_3 + u_2 v_1}{\sqrt{u_1^2 + v_1^2}, \quad \sin \theta = \frac{u_2}{\sqrt{u_1^2 + v_1^2}}, \quad \sin \phi = \frac{u_1}{\sqrt{u_2^2 + v_2^2}}
\]
to obtain the formula.

Remark on cross-ratios

a) Let 5, x, a, b be four concurrent lines.
   The cross-ratio is defined by
   \[(S:ta:5:ab) = (S:ta:5:ab)\]
   where 5, t, a, b are the points the lines meet at a fifth line not concurrent with the four.

b) The cross-ratio is a projective invariant.

2) Let \(l_1, m_1, m_2\) be the
   images of lines \(l, m\)
   in \(x\) meeting at \(P\).
   Let 5, x be the tangents
to the absolute from the
   points \(P_1\) of intersection
   of \(l_1\) and \(m_1\), lines of \(P\).
   The lines 5, x, \(l_1, m_1\),
   are concurrent, but 5, x are non-parallel lines.
   Then the NC angle \(\gamma\) between \(l, m\) at \(P\)
   is given by
   \[\gamma = \pm \frac{1}{2 \tan^{-1} \log (s + t + u)}\]

Proof. Connecting
Order formulas

1) Let \( u, x + v, y - w = 0 \)
\[ u_x + v_y - w = 0 \]
be the images of non-intersecting lines in \( x \).

The NE distance of between these lines in \( x \) (a change of unique common perpendicul overthrow) is given by:

\[
\text{cosh} \frac{d}{h} = \pm \frac{\sqrt{u_y^2 + v_x^2} - w}{\sqrt{u_x^2 + v_y^2} - w^2}
\]

2) Let \( (x_0, y_0) \)
and \( u_x + v_y - w = 0 \)
be the images of a point and a line in \( x \).

The NE distance of between the point and line is given by:

\[
\sinh \frac{d}{h} = \pm \frac{\sqrt{u_y^2 + v_x^2} - w^2}{\sqrt{u_x^2 + v_y^2} - w^2}
\]

Remark: The above three formulas can also be rewritten in terms of cross-ratios.

Remark: Using coordinates \( x \) and \( y \) as above on the complex plane, it is easy to check circles, branches, and equivalent curves have on their images curves in the \( x \) and \( y \) coordinates.

circle  
hyperbola  
equidistant curves
Beltrami's model

Define a fixed circle \( x^2 + y^2 = r^2 \) in the Euclidean plane as the absolute.
Interpret the primitive notions of NE geometry as follows:

1) point: point interior to the circle
2) line: chord of the circle (less end points)
3) arc and between: usual way

Define distance between points by

\[
d = \frac{\sqrt{r^2 - x_1^2 - y_1^2}}{\sqrt{r^2 - x_2^2 - y_2^2}} \quad (r = h)
\]

and angle between lines by

\[
cos \phi = \pm \frac{d_1^2(x_1y_2 + x_2y_1) - d_2^2x_1y_2}{\sqrt{d_1^2(x_1^2 + y_1^2) - x_1^2} \sqrt{d_2^2(x_2^2 + y_2^2) - x_2^2}}
\]

Interpret:

4) congruent: for segments by equal length
5) congruent: for angles by equal angle

Next: Verify that all the axioms of NE geometry hold for this interpretation. Requires some effort.
Result: 1) NE geometry is consistent! If a contradiction is found in NE geometry, there is already one in Euclidean geometry, and the above model is consistent in Euclidean geometry. But: NE Euclidean geometry is modeled on anti-Euclidean (real number) and Cartesian coordinates, so is consistent if and only if consistent.

Note: Earlier work: Euclidean geometry is modeled on NE geometry such as a horosphere, so one get Euclidean geometry is consistent if NE geometry is consistent.
2) The Fifth Postulate of Euclid is independent: It is impossible to remove it from the other axioms, as it or its negation, the NE Axiom, may be assumed, and a consistent set of axioms result.
Cayley-Klein model

Designate a fixed
circle in the real
projective plane as
the absolute.
Interpret the primitive
entities of NE geometry
as follows:
1) point = point interior to the circle.
2) line = chord of circle (less end points).
3) area and batteries = usual away.
Define distance between points by
\[ d = \pm \frac{1}{2} \log \left( \frac{P Q}{17 N N} \right) \]

and angle between lines by
\[ \theta = \pm \frac{1}{2 \pi} \log \left( \frac{5 \times 10^4}{\mu} \right) \]

Interpret:
4) congruent: for segments by equal lengths.
5) congruent: for angles by equal angles.

Next: Verify that all the axioms of NE geometry hold for this interpretation. Requires some effort, and techniques of projective geometry.
Result: 1) NE geometry is consistent - another consequence of consistency of projective geometry regarded on earth mere.
2) F. P. Postulate of Euclid in contradiction.

Note: Euclidean geometry may also be modeled as projective geometry by designating a line to be at infinity.
Inversion:
In the Euclidean plane, inversion with respect to a circle of center O and radius r
is defined by \( OP \cdot OP' = r^2 \), and \( O, P, P' \) collinear.
Under inversion, circles in general correspond to circles, but a line through O correspond to
straight lines.
1. Inversion is conformal - angles between curves are preserved.
2. If \( P \) and \( Q \) correspond to \( P' \) and \( Q' \), then
\( \triangle POQ \sim \triangle Q'O'P' \) (similar triangles).

Cross ratio:
For four points \( P, N, A, B \) on a circle, define the cross ratio

\[ (MNAB) = \frac{MA}{NA} \cdot \frac{NB}{BA} \]

whence \( MA, NB \) denote chord lengths (7 to 8, res signs).

Using 2) above, one
shows: If \( P, N, A, B \)
on one side
are inverse
correspond to \( P', N', A', B' \)
on another side
under inversion,
then

\[ (MNAB) = (MN'AB') \]

Thus: The cross ratio of inversion

\[ (MNAB) \]
is unchanged under inversion with respect
to various circles - an inversion is invariant.
First model. Ref. Text Ch VIII for full discussion.

Preliminaries

In Euclidean space, let $\beta$ be a plane containing Euclidean's model. Let a sphere cut this plane along its axis of symmetry (a circle), and let $N$ be a pole of this circle.

Transform the midcircle of the circle to itself in two steps:

1) Stereographic projection from $N$: $N$, $P_1$, $P_i'$ are spheres.
2) Stereographic projection from $N'$: $NP_i''$, center $N'$ is $P_i''$.

Then: The image of a chord is an arc of a circle orthogonal to the circle whose axis is the abscissa.

Interesting lines. Parallel lines. Non-intersecting lines.

Distance. Let the circle orthogonal to the abscissa and center $P_i'$, $Q_i'$
the image of $P_i'$, $Q_i'$, meet the abscissa in $M_i$, $N_i$.

Then the NE distance $d = P_i'Q_i'$ is given by:

$$d = \pm h \log (MN_iP_i'Q_i').$$
Proof. 1) We show \((MNP, QA) = (MNP', QA')\), and then

assertion follows from previous distance formula using well-known facts. This equality is proved by

foot motion in case of a line through 0.

2) In the 3D plane,

let \(OC \perp \text{line } PQ\),

and on line \(OC\) perform vector \(OC = OC'\),

so \(OC = A, B\) on \(OA = CP\),

\(OB = CQ\). Then \(AB = PQ\). Draw lines parallel to

lines \(AB\) and \(PQ\). The line

perpendicular to \(OC\) at its midpoint \(O\) is line of symmetry of this figure (F).

3) The image of (F) in

the Beltrami model is

as shown. The

plane of lines parallel

to one line of symmetry

have images meeting

in \(V\). Clearly:

\((MNP, QA) = (MNP', QA')\).

4) The image of (F) under the transformation above is

as shown. - (F').
We locate the line as a symmetry in \((L')\) by a circle so that six vertices in this circle create \(MN, N, P', Q',\) and \(L, K, A', B'.\) Then we will have \((MNP', Q') = (KL A', B')\) and cross notes on circles.

**Note:** Orient families and subscripts for convenience.

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Let \(V\) be diametrical opposite \(C\) on circle \(MN.\) Construct a circle with center \(V\) and radius \(s,\) \(s^2 = OV \cdot CV.\)

**Note:** \(OV \cdot OC = s^2\) and circle \(MN\) is orthogonal to the circle \(V\) at \(C\) if tangent to \(MN\) at \(C\) and \(OV = s.\)

**Note:** \(CV = OV - OC\) so \(s^2 = OV \cdot (OV - OC) = OV^2 - OV \cdot OC = OV^2 - s^2.\) The circle \(V\) is orthogonal to the circle \(MN\) at \(C\).

**Note:** Suppose we write \(V\) under \(C\) to \(O\) by \(OV \cdot CV = s^2,\) it results that \(MN\) is orthogonal to the circle \(V.

**Note:** Inversion in circle \(V:\)

i) Preserve the absolute - since \(V\) is orthogonal to circle \(V.

ii) \(s\) and \(C\) to \(O\) - since \(OV \cdot CV = s^2.\)

iii) Simultaneously \(MN,\) circle is orthogonal to the absolute and \(OC\) at \(C,\) into a circle orthogonal to the absolute and \(OC\) at \(O.\) - Necessity line \(KL.\)

iv) Inversion in circle \(V\) moves \(MN\) to \(KL\) and then clearly \(P, Q\) to \(A, B\) as desired.
4) We need and easy shows: In one case of a line passing through 0, \( (KL, A, B) = (KL, A', B') \).

Let \( a = 0A, \ b = 0B, \ a' = 0A', \ b' = 0B' \).

From the figure given:
1. Transformation and similar triangles:
   \[
   \frac{a'}{a} = \frac{a - a'}{\sqrt{a^2 - a'^2}}.
   \]
2. Solve for \( a \):
   \[
   a = \frac{2a^2a'}{a^2 + a'^2}.
   \]

Then:
   \[
   (KL, A, B) = \frac{\sigma + a}{\sigma - a} = \left( \frac{\sigma + a}{\sigma - a} \right)^2 = (KL, A', B')^2. \quad \text{QEP.}
   \]

Angles let \( l', m', n' \) be
the mirror of lines \( l, m \)
in a meeting in \( P \).

\( l', m' \) are circles orthogonal
at the absolute and meeting
at \( P' \). The NE angle of
between \( l, m \) at \( P \) is
given by:

\[
\theta = \text{Euclidean angle at } P', \text{ between curves } \]
\( l', m' \) as measured by tangents at \( P' \).

This feature is consistent to the Euclidean plane.

Proof: In a, construct new angle at \( O \) equal to \( \theta \) using perpendicular to \( OP \) at mid-point or side of symmetry. In the same figure, we get a new angle at \( O' \), by reflection in a circle or mirror plane proof, whereas is equal to the angle.
between $h_1$, $h_1$ at $p_1$, since miniscus is continuous.
Thus: it suffices to prove the assertion for
angles at $0$. Here it is obvious since an angle in
$a$ and $b$ are same at $0$. Q.E.D.

Note: See also Text §104, §114 ff.

Relative angles
to distance by obtaining the angle of parallelism
formula, making unit circles.

The model
designate a fixed
circle of radius $r = h$
as the absolute.
Interpret the primitive
notions of NE geometry
as follows:
1) fixed: point distance to circle.
2) line: arc of circle orthogonal to absolute.
3) arc and line: usual way.

Define distance by $d = h \log (\text{MNPQ})$ and
angle by $\theta = \text{Euclidean angle}$.

Interpret:
4) congruence: for segments by equal distance
5) congruence: for angles by equal angles.

Next: Verify all axioms of NE geometry based
for this interpretation. $R$ expresses some effort
and the techniques of miniscus geometry.
Result: Same as before.
Ref. Text §28 ff for details.

Arrows, circles, hypercycles, equidistant curves are
represented by circles in this model.
Second model - upper half plane.

Begin with the Beltrami model as before, but reflect the second, stereographic projection by stereographic projection from a point on the circle to a plane tangent to the sphere at the diametrically opposite point.

Observe: Images of circles are circles - considered to be lines, straight lines considered as straight lines.

Note: This model is conformal to the first and to the Euclidean plane.

Increasing lines. Parallel lines. Non-intersecting lines.

One builds an interpretation of the NE axioms as before. Results: Same as before.

Aside. Curves, curve arcs, representant curves are represented by curves as in this model.