9. Miscellaneous

Exercises of the NE Plane

Remark: The following exercises is formulated for the NE plane. Much loose points generally; evidently, much of it is true for the Euclidean plane.

Exercises

A non-increment (or rigid motion) of the NE plane is a transformation $\mathcal{G}$ of the plane onto itself such that Euclidean distance: if $p' = \mathcal{G}p, q' = \mathcal{G}q$, then $p'q' = pq$.

Remarks:
1. An increment must start with either.
   For if $P, Q, R$ are collinear, say $PR = PQ + QR$, then $P'R' = P'Q' + Q'R'$ so $P', Q', R'$ are collinear.
3. An increment cannot create or destroy, hence the barycenter, etc. It commutes from 1) and 2).
4. The product $\mathcal{G} \mathcal{H} (p' = \mathcal{H}(\mathcal{G}p))$ and inverses $\mathcal{G}^{-1}$ of increment are increments. All increments form a group $\mathcal{G}$, the increment group of the NE plane. Thus: if $\mathcal{G}$ is an increment, then $\mathcal{G}^{-1}$ is also.

Remark: An increment must be a bijection, hence non-collinear facet, is necessarily the identity.

Proof: If $\mathcal{G}$ fixes
   \[ A, B, C : \mathcal{G}A = A \text{ etc.,} \]
   then $\mathcal{G}$ cannot have
   \[ AB, BC, CA \text{ as common lines,} \]
   thus every point on these lines. Let $P$ be arbitrary. Say $AP$ intersects $BC$ in $D$. $\mathcal{G}$ fixes:
   \[ A, D, \text{ line } AD, \text{ and others every point of } \]
Reflections in lines

Let \( l \) be a line.

For any point \( P \), let \( PK \perp l \) at \( K \); hence \( PK \perp P'K \) so \( P'K = PK \).

Set \( \Phi P = P' \). Then \( \Phi \) is a transformation of the plane onto itself and \( \Phi^2 = \mathcal{I} \) (identity).

\( \Phi \) is the reflection in the line \( l \).

Proof: Easy to see:
\[ \triangle P'KL \cong \triangle PKL \text{ by SAS} \]
\[ \triangle P'LQ' \cong \triangle PQL \text{ by SAS} \]

\( \therefore P'O' = PQ \). QED

Note: \( \Phi \) is indifferent.

Converging, \( G \) and \( G_0 \) are congruent on the frame of the plane. i.e. \( G \) and \( G_0 \) on \( G \) a \( \Phi \) under \( \Phi P = Q \).

Proof: 1) Case of \( G \).
Let \( l \) be perpendicular to \( PQ \) at its midpoint.
Take \( \Phi = \Phi_0 \).

2) Case of \( G_0 \). Let \( M \) be midpoint of \( PQ \), \( \mathcal{I} \) is perpendicular to \( PQ \) at its midpoint of \( PQ \) and \( M \).
Take \( \Phi = \Phi_0 \Phi_0 \). QED.

Properties of reflections.

We enumerate the possibilities for a point \( \Phi_m \Phi_0 \). Proofs are easy, and like the Euclidean case.
1) Case 1 and \( m \) are intersecting in point \( P \).
Then \( \Phi_m \Phi_e \) is a rotation about \( P \) through angle \( \alpha \), where \( \alpha \) is the angle from \( l \) to \( m \).

Consequently, \( \Phi \) contains all rotations through all angles about one point.

2) Case \( l \) and \( m \) are parallel. Then \( \Phi_m \Phi_e \) is a horizontal displacement.

3) Case \( l \) and \( m \) are non-intersecting. Then \( \Phi_m \Phi_e \) is a translation along the common perpendicular to \( l \) and \( m \) through distance \( d \), where \( d \) is the distance from \( l \) to \( m \) along the common perpendicular.

N.B. The product of translations is not in general a translation, as in the Euclidean plane.

Consequence: Every isometry is the product of at most three reflections in lines.

Proof:
1) Easy: If an isometry fixes a point \( P_0 \), it is either a rotation about \( P_0 \) (2 reflections in lines) or a reflection in a line through \( P_0 \).
2) Fix \( P_0 \). Let \( \varphi \) in \( \Phi \). Let \( \psi \) in \( \Phi \) be a reflection in a line carrying \( P_0 \) to \( \varphi P_0 \). Then \( \psi^{-1} \varphi \) fixes \( P_0 \): \( \psi^{-1} \varphi P_0 = P_0 \). Thus

\[ \varphi = \psi \psi^{-1} \varphi \]

is a product of at most 2 reflections.

Remark: It follows that:
1) A product of an even number of reflections

...
is the equality or equal to the product of 2 reflections.

1) The product of an odd number of reflections is equal to the product of 1 or 3 reflections.

**Congruence**

**Theorem:** Two triangles are congruent: \( \triangle ABC \cong \triangle A'B'C' \), exactly means there is a rigid motion \( \Phi \) carrying one onto another: \( \Phi A = A', \Phi B = B', \Phi C = C' \).

**Remark:** Congruence is transitive.

**Proof:** Let \( \Psi_1 \) take \( A \) to \( A' \). Let \( \Psi_2 \) be a rotation about \( A' \) on line from \( A' \) to \( B \), \( \Psi_2 B \) is carried to line \( A'B' \). Then \( \Psi_2 \Psi_1 B = B' \).

Either \( \Psi_2 \Psi_1 C = C' \) and one take \( \Phi = \Psi_2 \Psi_1 \) on \( \Psi_2 \Psi_1 C \) and \( C' \) are on opposite sides of line \( AB \) and one take \( \Phi = \frac{1}{2} \text{line } AB \).

2) If \( \Phi \) and \( \Phi' \) are linear such is commutative, \( \Phi' \Phi \) fixed \( A, B, C \) is \( \Phi' \Phi = I \).

**Problem:** Show: The product \( \Phi \Phi' \Phi'' \) of three reflections in lines is a reflection in a line of and only if the three lines \( l, m, n \) belong to a plane. Note: This holds in our PE and Euclidean frames.

The Weingarten model

Let \( r \) and \( \theta \) be polar coordinates in the NE plane, \( r = 0 \)

being the origin \( O \).

Set

\[
\begin{align*}
x &= r \sin \theta \frac{E}{h} \\
y &= r \cos \theta \frac{E}{h} \\
z &= r \cos \theta \frac{F}{h}
\end{align*}
\]

Then - easy. \( x^2 + y^2 - z^2 = h^2 \). The NE plane is modeled on the sheet \( z > h \) of a hyperbolic sheet of two sheets - a "sphere" or Heineken's cone. \( O \) has coordinates \((0, 0, h)\).

Note: Heineken's cone has vertex coordinates \((r, 0, 0)\).

The square length of a vector is measured by \( x^2 + y^2 - z^2 \), and this comes from the inner product \( x_1 x_2 + y_1 y_2 - z_1 z_2 \).

The distance formula

Let \((x_1, y_1, z_1), (x_2, y_2, z_2)\) represent two points of the NE plane. Then the NE distance between the points is given by

\[
\cosh \frac{d}{h} = \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2} \sqrt{x_2^2 + y_2^2 + z_2^2}}
\]

Remark: Compare analogous formula on a sphere of radius \( R \) in Euclidean space:

\[
\cos \frac{d}{R} = \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2} \sqrt{x_2^2 + y_2^2 + z_2^2}}
\]

Proof. Use coordinates \( r_1 \) and \( r_2 \) for the two points. From law of cosines:

\[
\cosh \frac{d}{h} = \cosh \frac{r_1}{h} \cosh \frac{r_2}{h} - \sinh \frac{r_1}{h} \sinh \frac{r_2}{h} \cos (\theta_1 - \theta_2).
\]

Expand \( \cos (\theta_1 - \theta_2) = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \).
\[
\begin{aligned}
\cosh \frac{a}{h} &= \cosh \frac{x_1}{h} \cosh \frac{x_2}{h} - \sinh \frac{x_1}{h} \sinh \frac{x_2}{h} \\
&= \frac{1}{h^2} (n_1 n_2 - n_3 n_4 - n_5 n_6) \\
&= \frac{1}{h^2} (x_1 y_2 - x_2 y_1 - y_2 x_1).
\end{aligned}
\]

The Lorentz group

1) A 3 \times 3 matrix \( H \) Preserve the pseudoform
\[ x'^2 + y'^2 - z'^2 = x^2 + y^2 - z^2, \]
where \( \left( \begin{array}{c} x' \\ y' \\ z' \end{array} \right) = H \left( \begin{array}{c} x \\ y \\ z \end{array} \right) \)

Such an \( H \) also preserves the spinor product
\[ x', y', z' = x, y, z. \]

2) A matrix \( H \) of the Lorentz group remains the hyperboloid of two sheets \( x'^2 + y'^2 - z'^2 = -1 \)
into itself. If \( H \) preserves "time", each sheet
is sent onto itself - the sheets are not
interchanged. Any subgroup of such \( H \) is
the reducible Lorentz group.

3) From the existence formulas above, every \( H \)
in \( \mathcal{L} \) gives an isometry of the NE plane
when the latter is represented by the Weyl-Hull
model. If \( H \) fixes every point of the
sheet \( 2\mathbb{R} \) of the hyperboloid \( x'^2 + y'^2 - z'^2 = -1 \),
then necessarily \( H = 3 \times 3 \) Lorentz matrices.

For item 1) from those invariance vector fields.

Composing \( \mathcal{L} \) is a subgroup of \( \mathcal{G} \).

\( \diamond \) The homomorphism from \( H \) to \( \mathcal{G} \) is \( 1 \)-
injective by last remark.
The matrices $M$ are characterized by

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$
Theorem: The group of isometries $G$ of the NE plane is isomorphic to the Euclidean isometry group $L$ if $G = L$.

Proof: We know $L$ is contained in $G$.

1) The matrices
$$R_x = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ in } L$$
gives a rotation of the NE plane about $O = (0, \theta)$ through an angle of $\alpha$.

The matrices
$$S_x = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ in } L$$
gives a reflection in the line through $O$ making an angle $\theta$ with the $x$-axis.

For $(\cos \theta, \sin \theta)$ and $(-\sin \theta, \cos \theta)$ under $(\cos \theta, \sin \theta)$ and $(-\sin \theta, \cos \theta)$ are fixed and reversed respectively.

Thus: $L$ contains all isometries in $G$ except for $0$.

2) The matrices
$$T_r = \begin{pmatrix} \cos \frac{r}{h} & 0 & \sin \frac{r}{h} \\ 0 & 1 & 0 \\ -\sin \frac{r}{h} & 0 & \cos \frac{r}{h} \end{pmatrix} \text{ in } L$$

where
$$0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad T_r O = \begin{pmatrix} h \sin \frac{r}{h} \\ 0 \\ h \cos \frac{r}{h} \end{pmatrix}$$

where $O$ is the point of the NE plane with

further coordinates $(r, 0)$. Hence $R \circ T_r$

where $O$ is

$$R \circ T_r O = \begin{pmatrix} h \sin \frac{r}{h} & 0 & 0 \\ 0 & 1 & 0 \\ h \cos \frac{r}{h} & 0 & 0 \end{pmatrix}$$

where $O$ is the point of the NE plane with

further coordinates $(r, 0)$.

Thus: $L$ is isometric on the points of the NE plane. For $2\pi$ and $\theta$ to $Q$, and $\theta$ to $O$, we have $O \to Q$.

3) To show $G = L$, one need show only every $\theta$ in $G$ also is in $L$, but $\theta$ in $G$. 

By (3) there is \( \mathbb{H} \) in \( \mathbb{R}^4 \) with \( \mathbb{H} \cap \mathbb{R} = \{ 0 \} \).
\[ \mathbb{H} \cap \mathbb{R}^4 = \{ 0 \} \]
by (1). Then \( \mathbb{H} = \mathbb{R}^4 \). \( \square \)

Remark: The term "components" of \( \mathbb{R}^4 \)

are known in the reasoning, correspond
to the proper and improper isometries of
the \( \mathbb{R}^4 \) plane.

In a completely analogous way, 3-dimensional
NE space has as a model the sheet \( \{ 0 \} \times (\mathbb{R}^3) \)
of \[ x^2 + y^2 + z^2 - t^2 = -1 \]
in 4-dimensional Minkowski space, presents form
\[ x^2 + y^2 + z^2 - t^2 = 0 \]. The graph of isometries of
NE space is reminiscent to the development velocity

larity graph of Minkowski space-time.

Thus: The "strange space universe" of
Eddington and Lobachevsky resembles the
"strange space universe" of Minkowski,
larity, and Einstein.