9. Miscellaneous

Isometries of the NE plane

Remark The following discussion is formulated for the NE plane. Much holds quite generally; especially, much of it is true for the Euclidean plane.

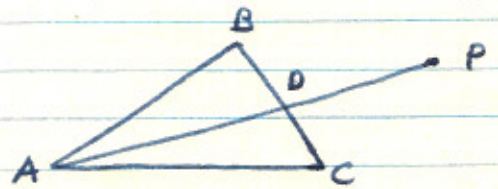
Isometries

An isometry (or rigid motion) of the NE plane is a transformation Φ of the plane onto itself which preserves distance: If $P' = \Phi P$, $Q' = \Phi Q$, then $P'Q' = PQ$.

Remarks 1) An isometry sends lines into lines. For if P, Q, R are collinear, say $PR = PQ + QR$, then $P'R' = P'Q' + Q'R'$ so P', Q', R' are collinear.
2) An isometry preserves angles. Immediate from Law of Cosines.
3) An isometry sends circles to circles, horocycles to horocycles, etc. Immediate from 1) and 2).
4) The products $\Psi\Phi$ ($P' = \Psi(\Phi P)$) and inverses Φ^{-1} of isometries are isometries. All isometries form a group G, the isometry group of the NE plane. The proper isometries, those (cont'd) →

Remark An isometry which fixes three non-collinear points is necessarily the identity.

Proof If Φ fixes A, B, C : $\Phi A = A$ etc, then Φ sends lines AB, BC, CA to themselves, and then fixes each point on these lines. Let P be arbitrary. Say AP meets BC in D. Φ fixes: A, D, line AD, and thus every point of

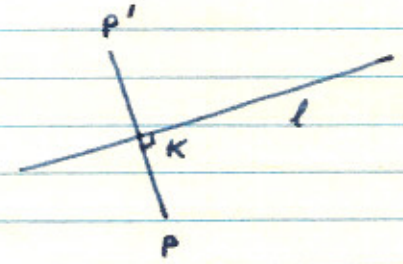


not necessarily, not reverse orientation, for a subgroup G_0 of index 2. I conjecture not in G_0 are <u>simple</u>.

line AD. Φ fixes P: $\Phi P = P$. QED.

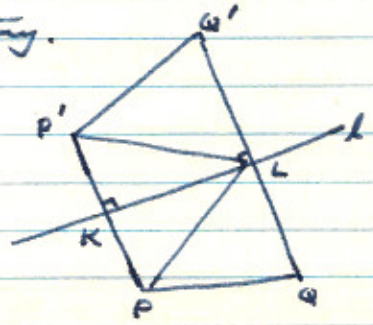
<u>Reflections in lines</u>

Let l be a line.
For any point P, let $PK \perp l$ at K; produce PK to P' so $P'K = PK$.
Set $\Phi_l P = P'$. Then Φ_l is a transformation of the plane onto itself and $\Phi_l^2 = I$ (identity).
Φ_l is the reflection in the line l.



<u>Claim</u> Φ_l is an isometry.

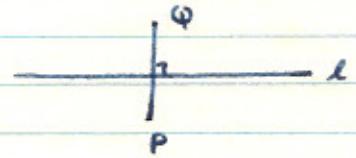
<u>Proof</u> Easy to see:
$\Delta P'KL \cong \Delta PKL$ by SAS
$\Delta P'LQ' \cong \Delta PLQ$ by SAS
so $P'Q' = PQ$. QED.



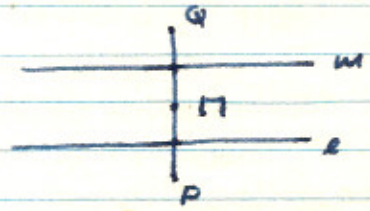
<u>Note</u> Φ_l is <u>improper</u>.

6/4 <u>Consequence</u> G and G_0 are transitive on the points of the plane. i.e. Given P, Q there is a Φ with $\Phi P = Q$.

<u>Proof</u> 1) Case of G.
Let l be perpendicular to PQ at its midpoint.
Take $\Phi = \Phi_l$.



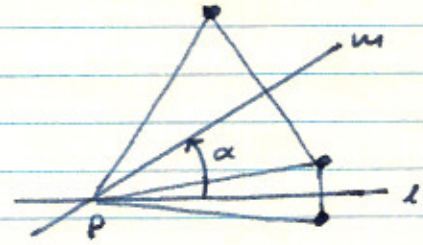
2) Case of G_0. Let M be midpoint of PQ, l, m perpendicular to PQ at midpoints of PM and MQ. Take $\Phi = \Phi_m \Phi_l$. QED.



<u>Products of reflections</u>.

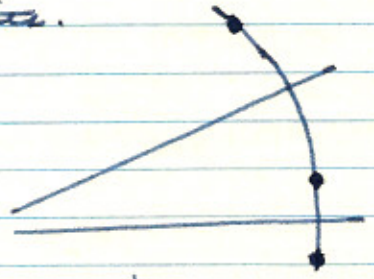
We enumerate the possibilities for a product $\Phi_m \Phi_l$. Proofs are easy, and like the Euclidean case.

1) Case l and m are intersecting in point P. Then $\Phi_m \Phi_l$ is a rotation about P through angle 2α where α is the angle from l to m.

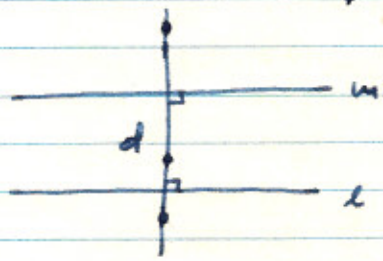


Consequence G_0 contains all rotations through all angles about all points.

2) Case l and m are parallel. Then $\Phi_m \Phi_l$ is a horocycle displacement.



3) Case l and m are non intersecting. Then $\Phi_m \Phi_l$ is a translation along the common perpendicular to l and m through distance $2d$ where d is the distance l to m along the common perpendicular.



N.B. The product of translations is not in general a translation as in the Euclidean plane.

Consequence Every isometry is the product of at most three reflections in lines.

Proof 1) Easy: If an isometry fixes a point P_0, it is either a rotation about P_0 (2 reflections in lines) or a reflection in a line through P_0.

2) Fix P_0. Let Φ in G. Let Ψ in G be a reflection in a line carrying P_0 to ΦP_0: $\Psi P_0 = \Phi P_0$. Then $\Psi^{-1}\Phi$ fixes P_0: $\Psi^{-1}\Phi P_0 = P_0$. Thus

$$\Phi = \underbrace{\Psi}_{\text{1 reflection}} \underbrace{\Psi^{-1}\Phi}_{\text{product of at most 2 reflections}}$$

QED.

Remark It follows that:

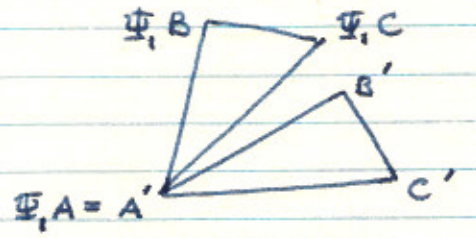
1) A product of an even number of reflections

is the identity or equal to the product of 2 reflections.
2) A product of an odd number of reflections is equal to the product of 1 or 3 reflections.

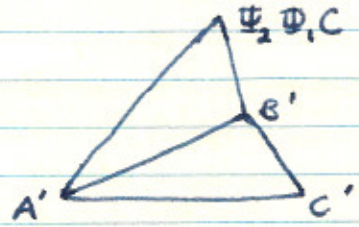
Congruence

Intuitively: Two triangles are congruent: $\Delta ABC \cong \Delta A'B'C'$, exactly when there is a rigid motion Φ carrying one onto the other: $\Phi A = A'$, $\Phi B = B'$, $\Phi C = C'$.

Theorem. If $\Delta ABC \cong \Delta A'B'C'$, then there is a unique isometry Φ so that $\Phi A = A'$, $\Phi B = B'$, $\Phi C = C'$.
Remark Converse is trivial.



Proof 1) Let Ψ_1 take A to A'. Let Ψ_2 be a rotation about A' so line from A' to $\Psi_1 B$ is carried to line $A'B'$. Then $\Psi_2 \Psi_1 B = B'$. Either $\Psi_2 \Psi_1 C = C'$ and we take $\Phi = \Psi_2 \Psi_1$ or $\Psi_2 \Psi_1 C$ and C' are on opposite sides of line AB and we take $\Phi = \Phi_{\text{line } AB} \Psi_2 \Psi_1$.

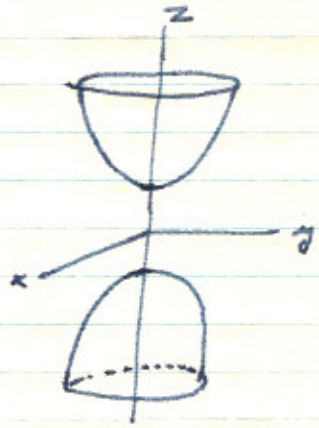


2) If Φ and Φ' are two such isometries, $\Phi^{-1}\Phi'$ fixes A, B, C so $\Phi^{-1}\Phi' = I$. QED

Problem 20 Show: The product $\Phi_n \Phi_m \Phi_l$ of three reflections in lines is a reflection in a line if and only if the three lines l, m, n belong to a pencil. Note: This holds in the NE and Euclidean planes.

Ref. H.W. Guggenheimer. Plane Geometry and its Groups, Holden-Day, San Francisco, 1967. Exp. p. 281.

The Weierstrass model



Let r and θ be polar coordinates in the NE plane, $r=0$ being the origin O.
Set

$$x = h \sinh \frac{r}{h} \cos\theta$$
$$y = h \sinh \frac{r}{h} \sin\theta$$
$$z = h \cosh \frac{r}{h} .$$

Then - easy - $x^2+y^2-z^2=-h^2$. The NE plane is imbedded as the sheet $z \geq h$ of a hyperboloid of two sheets - a "sphere" in Minkowski space. O has coordinates $(0,0,h)$.
Note Minkowski space has three coordinates x, y, z. ~~Lengths are measured by the square~~ The square length of a vector is measured by $x^2+y^2-z^2$, and this comes from the inner product $x_1x_2+y_1y_2-z_1z_2$.

The distance formula

Let (x_1, y_1, z_1), (x_2, y_2, z_2) represent two points of the NE plane. Then the NE distance d between these points is given by

$$\cosh \frac{d}{h} = \frac{z_1z_2 - x_1x_2 - y_1y_2}{\sqrt{z_1^2-x_1^2-y_1^2}\sqrt{z_2^2-x_2^2-y_2^2}} \qquad \left(\begin{array}{c}\text{Denominator}\\ \text{is } h\cdot h.\end{array}\right).$$

Remark Compare analogous formula on a sphere of radius R in Euclidean space:

$$\cos \frac{d}{R} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2+y_1^2+z_1^2}\sqrt{x_2^2+y_2^2+z_2^2}} .$$

Proof Use subscripts 1 and 2 for the two points. From Law of Cosines:

$$\cosh \frac{d}{h} = \cosh \frac{r_1}{h} \cosh \frac{r_2}{h} - \sinh \frac{r_1}{h} \sinh \frac{r_2}{h} \cos(\theta_1-\theta_2).$$

Expand $\cos(\theta_1-\theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$:

$$\cosh\frac{d}{h} = \cosh\frac{r_1}{h}\cosh\frac{r_2}{h} - \sinh\frac{r_1}{h}\cos\theta_1 \sinh\frac{r_2}{h}\cos\theta_2$$
$$- \sinh\frac{r_1}{h}\sin\theta_1 \sinh\frac{r_2}{h}\sin\theta_2$$
$$= \frac{z_1}{h}\frac{z_2}{h} - \frac{x_1}{h}\frac{x_2}{h} - \frac{y_1}{h}\frac{y_2}{h}$$
$$= \frac{1}{h^2}(z_1 z_2 - x_1 x_2 - y_1 y_2).$$ QED.

<u>The Lorentz group</u>

1) A 3×3 matrix M <u>preserves</u> the quadratic form $x^2+y^2-z^2$ if $x'^2+y'^2-z'^2 = x^2+y^2-z^2$, where $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = M \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, for all x, y, z.

Such an M also preserves the inner product:
$$x_1'x_2' + y_1'y_2' - z_1'z_2' = x_1x_2 + y_1y_2 - z_1z_2.$$
All such matrices M alone constitute a group, the <u>full Lorentz group</u>. (QED) →

6/6 2) A matrix M of the Lorentz group sends the hyperboloid of two sheets $x^2+y^2-z^2 = -h^2$ into itself. If M preserves "time", each sheet is sent into itself – the sheets are <u>not</u> interchanged. The subgroup of such M is the <u>isochronous</u> Lorentz group, denoted by L. L has index 2 in the full Lorentz group. L has two "components" – according as "<u>time</u>" is reversed by M or not.

3) From the distance formula alone, every M in L gives an isometry of the NE plane when the latter is represented by the Weierstrass model. If M fixes every point of the sheet $z \geq h$ of the hyperboloid $x^2+y^2-z^2=-h^2$, then necessarily $M =$ 3×3 identity matrix. (For then M fixes three independent vectors.)

<u>Consequence</u> L is a subgroup of G.
i.e. The homomorphism from L to G in 3) is injective by last remark.

These matrices M are characterized by

$$^tMJM = J \quad \text{where} \quad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

<u>Theorem</u> The group of isometries G of the NE plane is isomorphic to the isochronous Lorentz group L : $G = L$.

<u>Proof</u>. We know L is contained in G.

1) The matrix
$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ in } L$$
gives a rotation of the NE plane about $O = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}$ through an angle of α.

The matrix
$$S_\alpha = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha & 0 \\ \sin 2\alpha & -\cos 2\alpha & 0 \\ 0 & 0 & +1 \end{pmatrix} \text{ in } L$$
gives a reflection in the line through O making an angle α with the $\theta = 0$ axis OX.

For $\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$ and $\begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}$ under $\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$ are fixed and reversed respectively.

Thus: L contains all isometries in G which fix O.

2) The matrix
$$T_r = \begin{pmatrix} \cosh\frac{r}{h} & 0 & \sinh\frac{r}{h} \\ 0 & 1 & 0 \\ \sinh\frac{r}{h} & 0 & \cosh\frac{r}{h} \end{pmatrix} \text{ in } L$$
sends
$$O = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} \text{ to } T_r O = \begin{pmatrix} h\sinh\frac{r}{h} \\ 0 \\ h\cosh\frac{r}{h} \end{pmatrix}$$
which is the point of the NE plane with polar coordinates $(r, 0)$. Hence $R_\theta T_r$ sends O to
$$R_\theta T_r O = \begin{pmatrix} h\sinh\frac{r}{h}\cos\theta \\ h\sinh\frac{r}{h}\sin\theta \\ h\cosh\frac{r}{h} \end{pmatrix}$$
which is the point of the NE plane with polar coordinates (r, θ).

Thus: L is transitive on the points of the NE plane. For to send P to Q, send P to O and then O to Q.

3) To show $G = L$, we need show only every Φ in G also is in L. Let Φ in G.

By 2) there is M in L with $MO = \Phi O$. $M^{-1}\Phi$ fixes O: $M^{-1}\Phi O = O$, so $M^{-1}\Phi$ is in L by 1). Then $\Phi = M . M^{-1}\Phi$ is in L. QED

Remark The two "components" of L, ~~time~~ space preserving or ~~time~~ space reversing, correspond to the proper and improper isometries of the NE plane.

In a completely analogous way, 3-dimensional NE space has as a model the sheet $x \geq h$ of $x^2 + y^2 + z^2 - x^2 = -h^2$ in 4-dimensional Minkowski space with quadratic form $x^2 + y^2 + z^2 - x^2$. The group of isometries of NE space is isomorphic to the inhomogeneous Lorentz group of Minkowski space-time. Thus: The "strange new universe" of Bolyai and Lobachevsky touches the "strange new universe" of Minkowski, Lorentz, and Einstein.