Examples of surfaces

One parameter families

Family \(\Sigma_1\) \(f(x, y, z, x) = 0\),
\(x = \text{parameter}\).

Intersection of 3 nearby surfaces:
\[ f(x, y, z, x) = 0 \]
\[ f(x, y, z, x + \Delta x) = 0 \]
\[ \frac{\Delta f}{\Delta x} = 0 \]

Let \(\Delta x \to 0\):

One parameter family with parameters 3.

These curves resemble envelopes of families.
\(x = \text{parameter}\) from \(f(x, y, z, x) = 0\).
\(x = a(x, y, z)\) where \(f_x(x, y, z, x) = 0\).
Envelopes: \((x, y, z, a(x, y, z)) = 0\).

Example: Sphere \(z = \sqrt{1 - (x-x_0)^2 + (y-y_0)^2}\)
\(f(x, y, z, x) = (x-x_0)^2 + y^2 + z^2 - 1\)
\(f_x(x, y, z, x) = 2(x-x_0)\)
\(f_x = 0 \Rightarrow x = a(x, y, z) = x\)
Envelopes: \(f(x, y, z, a(x, y, z)) = y^2 + z^2 = 1\)
c. The cylinder \(z = \sqrt{1 - y^2}\).

Tangency. Apart from singular points, this manifold \(S\) is tangent to each member of the family of surfaces along the corresponding characteristic curve \(c_x\).
For: let \((x, y, z)\) lie on the
\(x\) change curve \((\text{curve})\)
\(\Delta f(-x) = 0 \& f_x(-x) = 0\);
this curve lies on the \(x\)
surface. Normal direction to envelope \(c_x\)
\(f(x, y, z, a(x, y, z)) = 0\).
\(\left(\frac{df}{dx} + \frac{df}{dz} \frac{dz}{dx}, \frac{df}{dy} - \frac{df}{dx} \frac{dx}{dy}, \frac{df}{dz} + \frac{df}{dx} \frac{dx}{dz}\right)\).
The normal direction \((x, y, z, a, b)\) to \(f(x, y, z, x) = 0\).

Two parameter families:

Family \(f(x, y, z, x, \alpha, \beta) = 0\) \(\alpha, \beta\) parameters.

Intersection of nearby surfaces:
\[
\begin{align*}
\large f(x, y, z, x, \alpha) = 0 & \quad \frac{\partial f}{\partial \alpha} \bigg|_{\alpha = 0} = \frac{\partial f}{\partial x} \bigg|_{\alpha = 0} = 0 \quad \text{Characteristic} \\
\large f(x, y, z, x, \beta) = 0 & \quad \frac{\partial f}{\partial \beta} \bigg|_{\beta = 0} = \frac{\partial f}{\partial x} \bigg|_{\beta = 0} = 0 \quad \text{Tangent plane.}
\end{align*}
\]

Envelope is locus of characteristic points:
\[
\begin{align*}
\frac{\partial f}{\partial \alpha} \bigg|_{\alpha = 0} & = \frac{\partial f}{\partial x} \bigg|_{\alpha = 0} = 0 \Rightarrow x = a(x, y, z) \\
\frac{\partial f}{\partial \beta} \bigg|_{\beta = 0} & = \frac{\partial f}{\partial x} \bigg|_{\beta = 0} = 0 \Rightarrow \beta = b(x, y, z).
\end{align*}
\]

Example: Envelope is tangent to each member of the family of surfaces at a characteristic point.

\[
\begin{align*}
E_1 & \quad \text{Sphere} \quad z = \sqrt{1 - (x - \alpha)^2 - (y - \beta)^2} \\
& \quad f(x, y, z, x, \alpha, \beta) = (x - \alpha)^2 + (y - \beta)^2 + z^2 - 1 \\
& \quad \left\{ \begin{array}{l} 
& \quad \frac{\partial f}{\partial \alpha} = -2(x - \alpha) \Rightarrow \alpha = a(x, y, z) = x \\
& \quad \frac{\partial f}{\partial \beta} = -2(y - \beta) \Rightarrow \beta = b(x, y, z) = y
\end{array} \right.
\end{align*}
\]

Envelope:
\[
f(x, y, z, ax, ay, x) = z^2 - 1.
\]

Example: Sphere of fixed radius with center on a given surface.

Surface \(x = R(\alpha, \beta)\)
given by parameter \(\alpha, \beta\).
\[
f(x, \alpha, \beta) = \left( x - R(\alpha, \beta) \right)^2 - c^2
\]
- Sphere of radius \(c\).

\[
\begin{align*}
\frac{\partial f}{\partial \alpha} & = -2(x - R) \cdot \frac{\partial \overline{R}}{\partial \alpha} \\
\frac{\partial f}{\partial \beta} & = -2(x - R) \cdot \frac{\partial \overline{R}}{\partial \beta}
\end{align*}
\]

Let \(\overline{S}(\alpha, \beta)\) be chosen point of envelope corresponding to sphere with center \(R(\alpha, \beta)\).

\[
\left\{ \frac{\partial f}{\partial \alpha} = 0 \Rightarrow \overline{S} = \frac{\partial \overline{R}}{\partial \alpha} \right\} \quad \text{and} \quad \left\{ \frac{\partial f}{\partial \beta} = 0 \Rightarrow \overline{S} = \frac{\partial \overline{R}}{\partial \beta} \right\}
\]
\( \mathbf{N}(x, \beta) \) unit normal to \( \mathbf{R}(x, \beta) \) surface \( \mathbf{S} - \mathbf{R} = \lambda \mathbf{N} \), \( \lambda = \text{mean} \).  
\[ f = 0 \Rightarrow (\mathbf{S} - \mathbf{R}) - c^2 = 0 \Rightarrow \lambda = \pm c. \]

Envelope is parameter surface \( \mathbf{X} = \mathbf{S}(x, \beta) = \mathbf{R}(x, \beta) + c \mathbf{N}(x, \beta) \).

**Equations of PDEs.** Preliminary remarks.

The one-parameter family of surfaces \( u = \varphi(x, y, z) \) have envelope \( u = \varphi(x, y, \alpha(x, y)) \) where \( x = \alpha(x, y) \) and \( \varphi_x (x, y, \alpha) = 0 \).  
In this case \( f(x, y, z, x) = \varphi(x, y, \alpha) - z \).

**Solutions of PDEs.**

One parameter family of 

\[
\begin{align*}
  f(x, y, z, \alpha) &= \varphi(x, y, \alpha) - z \\
  f'_{\alpha} &= \varphi_x
\end{align*}
\]

Envelope:

\[ u = \varphi(x, y, \alpha(x, y)) \]

where \( x = \alpha(x, y) \) and \( \varphi_x = 0 \). \( \varphi_x (x, y, \alpha) = 0 \).  
Two parameter family:

\[ u = \varphi(x, y, x, \beta) \]

Envelope:

\[ \begin{align*}
  \varphi_x &= 0 \\
  \varphi_y &= 0 \Rightarrow \beta = \beta(x, y).
\end{align*} \]

PDE \( F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0 \).

Fact: The envelope of a family of solutions

\( u \) is a solution.

Analytically:

\[ u = \varphi(x, y, \alpha(x, y)). \]

\[ \frac{\partial u}{\partial x} = \varphi_x + \varphi_{\alpha} \frac{\partial \alpha}{\partial x} = \varphi_x (x, y, \alpha(x, y)) \]

\[ \frac{\partial u}{\partial y} = \varphi_y + \varphi_{\alpha} \frac{\partial \alpha}{\partial y} = \varphi_y (x, y, \alpha(x, y)). \]

So \( F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = F(x, y, \varphi_x, \varphi_y, \varphi_{\alpha}) = 0 \).

Geometrically:

A \( x \) the plane \( (x, \gamma, \varphi_x(x, \gamma)) \) the solution \( u = \varphi(x, y, \gamma) \) is tangent to a plane \( u' = \varphi(x, y, \gamma) = b(x, y, x_0, y_0) \) \( \varphi(x, y, \varphi_x, \varphi_y, b(x, y, x_0, y_0)) = 0 \).  
The envelope is tangent to the solution \( \varphi \) and \( u \) is tangent to the plane and is itself a solution.

L'Hopital: Two parameter case.
On a fixed surface, \( q(x, y, x) - u = 0 \)

Form a new curve:

\[
\begin{align*}
q(x, y, x) - u &= 0 \\
q_x(x, y, x) &= 0 \Rightarrow x = ax + y
\end{align*}
\]

For each \( x \), obtain a curve. To gather these gives a surface,

\[ u = q(x, y, ax + y), \text{ the same as before}. \]

The level curves

\[ q(x, y, ax + y) = 0 \]

describe the surface, one by one, enveloped by the level curves, \( q(x, y, x) = u \).
One parameter families - given by relations
\[ f(x, y, z, x, \beta) = 0 \quad \text{two param.} \]
\[ g(x, \beta) = 0 \quad \text{vector family} \]
\[ \text{where } g = 0 \text{ for } \beta = \beta(x), \quad \text{one-param.} \]
\[ f(x, y, z, x, \beta(x)) = 0. \]

\[ \frac{\partial}{\partial x} f(x, y, z, x, \beta(x)) = 0 \quad \Rightarrow \quad f_x + \beta \frac{\partial f}{\partial \beta} = 0, \]
\[ g(x, \beta) = 0 \quad \Rightarrow \quad \frac{\partial g}{\partial x} = -\frac{g_x}{\beta} \quad \Rightarrow \quad f_x \beta f_\beta - f_\beta f_x = 0. \]

Example:
Given family
\[ \{ z = z_0 + F(x, y) + G(y, z) \} \]
\[ F(\beta, \eta) = 0 \]
\[ \begin{cases} f(x, y, z, \beta) = f(x, y, z) = 0, \\ \frac{\partial f}{\partial x} = 0, \end{cases} \]
\[ f \text{, } f_y \text{, } f_z \text{, } f_\beta \text{, } f_\eta \text{, } f_{\beta \eta} \text{, } f_{\beta z} \text{, } f_{\eta z}. \]

Example: One param. family
\[ \{ z = z_0 + F(x, y) + G(y, z) \} \]
\[ F(\beta, \eta) = 0, \]
\[ \begin{cases} f(x, y, z, \beta) = f(x, y, z) = 0, \\ \frac{\partial f}{\partial x} = 0, \end{cases} \]
\[ f \text{, } f_y \text{, } f_z \text{, } f_\beta \text{, } f_\eta \text{, } f_{\beta \eta} \text{, } f_{\beta z} \text{, } f_{\eta z}. \]

Lines through a point:
\[ \begin{vmatrix} 2 & \beta & \eta \\ y & z & -1 \end{vmatrix} = (-F_\beta, -F_{\eta z} - F_\eta, F_{\eta z} - F_\eta). \]

Cartesian form of lines:
\[ (x, y, z) = (x_0, y_0, z_0) + \lambda (F_\beta, F_z, F_{z,0} + F_\eta F_\beta) \]
\[ (x, y, z) = (x_0, y_0, z_0) + \lambda (F_\beta, F_z, F_{z,0} + F_\eta F_\beta) \]
\[ F(\beta, \eta) = 0. \]
Intersection of two infinitely near planes:
\[
\begin{align*}
F_1(x, y, z) &= 0, \\
F_2(x, y, z) &= 0
\end{align*}
\]

From \( F_1 \, dx + F_2 \, dz = 0 \) \( (F_1 = \frac{\partial F}{\partial y}, \, \text{arc}) \), above vector bundle is \((F_1, F_2, \, k\, F_1 + z \, F_2)\).

Cone of lines
\[
(x, y, z) = (x_0, y_0, z_0) + x (F_1, F_2, k\, F_1 + z \, F_2)
\]
with \( F(k, z) = 0 \).

Homogeneous case:
\[
\begin{align*}
\{ u(x-x_0) + v(y-y_0) + w(z-z_0) &= 0, \\
g(u, v, w) &= 0 \text{ homogeneous}
\end{align*}
\]