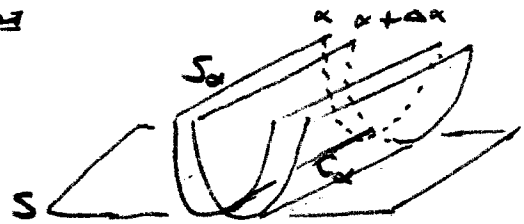
## Envelopes of surfaces

Smirnov p. 309 ff.

### One parameter families

Family $S_\alpha$: $f(x,y,z,\alpha)=0$,
$\alpha$ = parameter.




Intersection of nearby surfaces:

$$\begin{cases} f(x,y,z,\alpha)=0 \\ f(x,y,z,\alpha+\Delta\alpha)=0 \end{cases}$$

Equivalent to:

$$\begin{cases} f(x,y,z,\alpha)=0 \\ \dfrac{f(x,y,z,\alpha+\Delta\alpha)-f(x,y,z,\alpha)}{\Delta\alpha}=0 \end{cases}$$

Let $\Delta\alpha \to 0$:

$$C_\alpha \begin{cases} f(x,y,z,\alpha)=0 \\ f_\alpha(x,y,z,\alpha)=0 \end{cases}$$

One parameter family of curves in space. | Characteristic curves.

These curves sweep out envelope of family.

i.e. Eliminate $\alpha$ from $f=0$ & $f_\alpha=0$!

$\alpha = a(x,y,z)$ solves $f_\alpha(x,y,z,\alpha)=0$

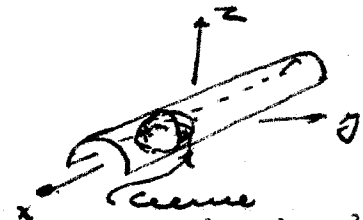
Envelope $S$: $f(x,y,z,a(x,y,z))=0$.

Eg spheres $z=\sqrt{1-(x-\alpha)^2+(y-0)^2}$

$f(x,y,z,\alpha) = (x-\alpha)^2+y^2+z^2-1$

$f_\alpha(x,y,z,\alpha) = -2(x-\alpha)$

$f_\alpha=0 \longrightarrow \alpha = a(x,y,z) = x$



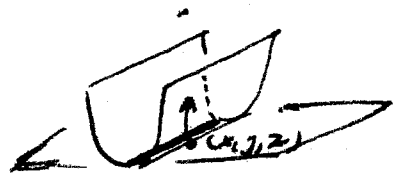

Envelope: $f(x,y,z,a(x,y,z)) = y^2+z^2-1$.

i.e. the cylinder $z=\sqrt{1-y^2}$.

Curve $\begin{cases} (x-\alpha)^2+y^2+z^2=1 \\ x=\alpha \end{cases}$

**Tangency.** Apart from singular points, the envelope $S$ is tangent to each member $S_\alpha$ of the family of surfaces along the corresponding characteristic curve $C_\alpha$.

For: Let $(x,y,z)$ lie on the $\alpha$ char. curve (namely is $f(-,\alpha)=0$ & $f_\alpha(-,\alpha)=0$); this curve lies on the $\alpha$ surface.




Normal direction to envelope at $f(x,y,z,a(x,y,z))=0$ is

$$\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial \alpha}\frac{\partial a}{\partial x},\ \frac{\partial f}{\partial y}+\frac{\partial f}{\partial \alpha}\frac{\partial a}{\partial y},\ \frac{\partial f}{\partial z}+\frac{\partial f}{\partial \alpha}\frac{\partial a}{\partial z}\right).$$

$f_\alpha = 0$ so this is the normal direction $(f_x, f_y, f_z)$ to $f(x, y, z, \alpha) = 0$.

<u>Two parameter families</u>

Family $f(x, y, z, \alpha, \beta) = 0$ $\quad \alpha$ & $\beta$ parameters

Intersection of nearby surfaces:

$$\begin{cases} f(x, y, z, \alpha, \beta) = 0 \\ f(\text{——}, \alpha + \Delta\alpha, \beta) = 0 \\ f(\text{——}, \alpha, \beta + \Delta\beta) = 0 \end{cases} \xrightarrow[\substack{\Delta\alpha \to 0 \\ \Delta\beta \to 0}]{\text{Let}} \begin{cases} f = 0 \\ f_\alpha = 0 \\ f_\beta = 0 \end{cases}$$

Characteristic points. Two param family.

Envelope is locus of characteristic points.

$$\begin{cases} f_\alpha(x, y, z, \alpha, \beta) = 0 \\ f_\beta(x, y, z, \alpha, \beta) = 0 \end{cases} \rightarrow \begin{matrix} \alpha = a(x, y, z) \\ \beta = b(x, y, z). \end{matrix}$$

Envelope is $f(x, y, z, a(x, y, z), b(x, y, z)) = 0$.

<u>Tangency</u>: Envelope is tangent to each member of the family of surfaces at a characteristic point.

<u>Eg.</u> Sphere $z = \sqrt{1 - (x-\alpha)^2 - (y-\beta)^2}$.

$f(x, y, z, \alpha, \beta) = (x-\alpha)^2 + (y-\beta)^2 + z^2 - 1$

$$\begin{cases} f_\alpha = -2(x-\alpha) \\ f_\beta = -2(y-\beta) \end{cases} \rightarrow \begin{matrix} \alpha = a(x, y, z) = x \\ \beta = b(x, y, z) = y \end{matrix}$$

Envelope:

$f(x, y, z, a(x, y, z), b(x, y, z)) = z^2 - 1$.

i.e. The planes $z = \pm 1$.

<u>Example</u> Spheres of fixed radius with centers on a given surface.

Surface $\vec{X} = \vec{R}(\alpha, \beta)$

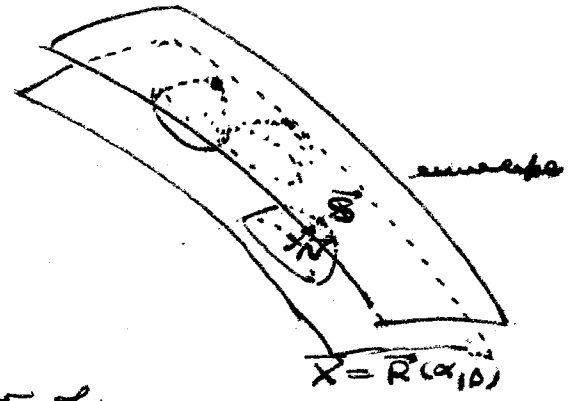
given by parameters $\alpha$ & $\beta$.

$f(\vec{X}, \alpha, \beta) = |\vec{X} - \vec{R}(\alpha, \beta)|^2 - c^2$

- sphere of radius $c$.

$f_\alpha = -2(\vec{X} - \vec{R}) \cdot \frac{\partial \vec{R}}{\partial \alpha}$

$f_\beta = -2(\vec{X} - \vec{R}) \cdot \frac{\partial \vec{R}}{\partial \beta}$




Let $\vec{S}(\alpha, \beta)$ be charac. point of envelope corresponding to sphere with center $\vec{R}(\alpha, \beta)$.

$$\begin{cases} f_\alpha = 0 \\ f_\beta = 0 \end{cases} \rightarrow \vec{S} - \vec{R} \perp \frac{\partial \vec{R}}{\partial \alpha} \text{ and } \frac{\partial \vec{R}}{\partial \beta}.$$

$\vec{N}(\alpha,\beta)$ unit normal to surface at $\vec{R}(\alpha,\beta)$. $\quad \vec{N} \perp \frac{\partial \vec{R}}{\partial \alpha}$ & $\frac{\partial \vec{R}}{\partial \beta}$.

So $\vec{S} - \vec{R} = \lambda \vec{N}$, $\lambda$ = scalar.

$f = 0 \rightarrow |\vec{S} - \vec{R}|^2 - c^2 = 0 \rightarrow \lambda = \pm c$.

Envelope is parametric surface:

$\vec{X} = \vec{S}(\alpha,\beta) = \vec{R}(\alpha,\beta) + c\,\vec{N}(\alpha,\beta)$.

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~~Solutions of PDEs. Preliminary remark.~~
~~The one-parameter family of surfaces~~
~~$u = \varphi(x,y,\alpha)$ have envelope $u = \varphi(x,y,a(x,y))$~~
~~where $\alpha = a(x,y)$ solves $\varphi_\alpha(x,y,\alpha) = 0$.~~
~~For this $f(x,y,z,\alpha) = \varphi(x,y,\alpha) - z$~~

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Solutions of PDEs.

One-param. family of surfaces $u = \varphi(x,y,\alpha)$.

$$\begin{cases} f(x,y,z,\alpha) = \varphi(x,y,\alpha) - z \\ f_\alpha = \varphi_\alpha \end{cases}$$

(OVER)

Envelope:

$u = \varphi(x,y,a(x,y))$

where $\alpha = a(x,y)$ solves ~~$\varphi_\alpha = 0$~~ $\varphi_\alpha(x,y,\alpha) = 0$.

Two parameter family:

$u = \varphi(x,y,\alpha,\beta)$

$$\begin{cases} \varphi_\alpha = 0 \\ \varphi_\beta = 0 \end{cases} \rightarrow \begin{array}{l} \alpha = a(x,y) \\ \beta = b(x,y). \end{array}$$

Envelope

$u = \varphi(x,y,a(x,y),b(x,y))$.

PDE $\quad F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right) = 0$.

Fact: The envelope of a family of solutions is a solution.

Analytically: $\quad u = \varphi(x,y,a(x,y))$.

$$\frac{\partial u}{\partial x} = \varphi_x + \varphi_\alpha \frac{\partial a}{\partial x} = \varphi_x(x,y,a(x,y)) \quad \& \quad \frac{\partial u}{\partial y} = \varphi_y.$$

$\varphi_\alpha = 0$

So $F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right) = F(x,y,\varphi,\varphi_x,\varphi_y) = 0$.

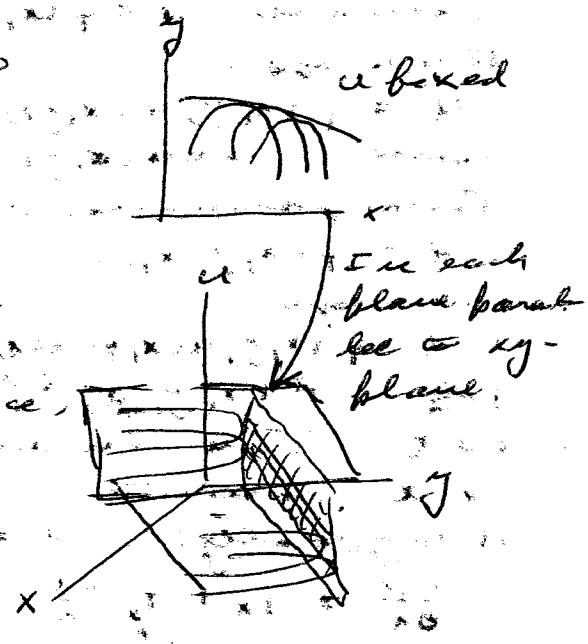
Geometrically: At the point $(x,y,\varphi(x,y))$ the sol'n $u = \varphi(x,y,\alpha)$ is tangent to a plane $u' - \varphi(x,y,\alpha) = p(x'-x) + q(y'-y)$ with $F(x,y,\varphi(x,y,\alpha),p,q) = 0$.
The envelope is tangent to the solution $\varphi$ and so is tangent to the plane and is itself a solution.

Likewise: Two parameter case.

Or Fix $u$, $\varphi(x,y,\alpha)-u=0$
a 1-param family of curves
Form envelope curve:

$$\begin{cases} \varphi(x,y,\alpha)-u=0 \\ \varphi_\alpha(x,y,\alpha)=0 \rightarrow \alpha=a(x,y) \end{cases}$$

obtain $\varphi(x,y,a(x,y))-u=0$.
For each $u$ obtain a curve.
To gather these give a surface,
$u=\varphi(x,y,a(x,y))$, the
same as before.




ie the level curves
$\varphi(x,y,a(x,y))=u$ of the
envelope are the envelopes
of the level curves $\varphi(x,y,\alpha)=u$.

<u>One parameter families</u> – given by relations

$$\left.\begin{array}{ll} f(x,y,z,\alpha,\beta)=0 & \text{two params.} \\ g(\alpha,\beta)=0 & \text{relation} \end{array}\right\} \rightarrow \text{one-param. family}$$

Solve $g=0$ for $\beta = h(\alpha)$, one-param. family is $f(x,y,z,\alpha,h(\alpha))=0$.

$$\left.\begin{array}{ll} \frac{\partial}{\partial\alpha} f(x,y,z,\alpha,h(\alpha))=0 \rightarrow f_\alpha + f_\beta \frac{dh}{d\alpha}=0 \\ g(\alpha,\beta)=0 \rightarrow \frac{dh}{d\alpha} = -\frac{g_\alpha}{g_\beta} \end{array}\right\} \rightarrow \begin{array}{l} f_\alpha - \frac{g_\alpha}{g_\beta} f_\beta = 0 \\ f_\alpha g_\beta - f_\beta g_\alpha = 0 \end{array}$$

Eliminate $\alpha$ and $\beta$.

$$\begin{cases} f(x,y,z,\alpha,\beta)=0 \\ g(\alpha,\beta)=0 \\ \frac{\partial(f,g)}{\partial(\alpha,\beta)}=0 \end{cases}$$

<u>Example</u> One param family of planes through a point.

$$\begin{cases} z-z_0 = p(x-x_0)+q(y-y_0) \\ F(p,q)=0 \end{cases}$$

$$\begin{cases} f(x,y,z,p,q) = p(x-x_0)+q(y-y_0)-(z-z_0) \\ F(p,q)=0 \\ f_p F_q - f_q F_p = (x-x_0)F_q - (y-y_0)F_p . \end{cases}$$

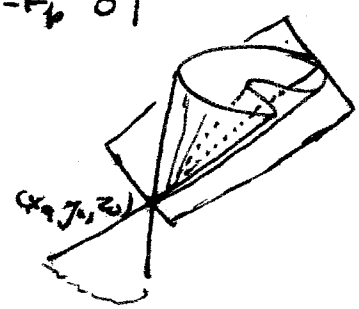
1st & 3rd eqns are two planes meeting in a line. Direction vector of line:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p & q & -1 \\ F_q & -F_p & 0 \end{vmatrix} = (-F_p, -F_q, -pF_p - qF_q).$$

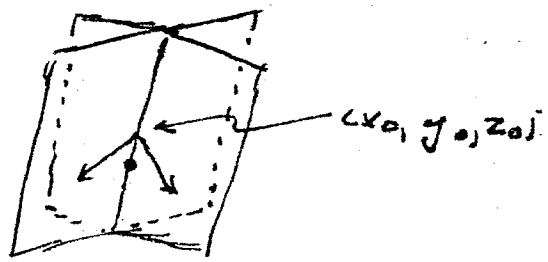
Cone of lines (generators):

$$(x,y,z) = (x_0,y_0,z_0) + t(F_p, F_q, pF_p+qF_q)$$

with $F(p,q)=0$.



~~Envelope of tangent~~
~~Inhomogeneous case.~~




Envelope of planes.

$$\begin{cases} z - z_0 = p(x-x_0) + q(y-y_0) \\ F(p,q) = 0 \end{cases}$$

One parameter family of planes through the point $(x_0, y_0, z_0)$.

Intersection of two infinitesimally near planes:

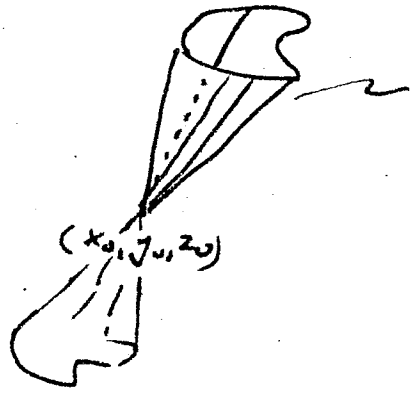
$$\begin{cases} p(x-x_0) + q(y-y_0) - (z-z_0) = 0, & F(p,q) = 0 \\ (p+dp)(x-x_0) + (q+dq)(y-y_0) - (z-z_0) = 0, & F(p+dp, q+dq) = 0 \end{cases}$$

is line with direction

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p & q & -1 \\ p+dp & q+dq & -1 \end{vmatrix} = (dq, -dp, p\,dq - q\,dp).$$

From $F_p\,dp + F_q\,dq = 0$ $\left(F_p = \frac{\partial F}{\partial p}, \text{etc}\right)$,

above vector parallel to $(F_p, F_q, pF_p + qF_q)$.




Cone of lines

$$(x,y,z) = (x_0, y_0, z_0) + t\,(F_p, F_q, pF_p + qF_q)$$

with $F(p,q) = 0$.

~~Homogeneous case~~

$$\begin{cases} u(x-x_0) + v(y-y_0) + w(z-z_0) = 0 \\ g(u,v,w) = 0 \quad \text{homogeneous} \end{cases}$$