Method of characteristics – Cauchy

F. John p. 19 ff.

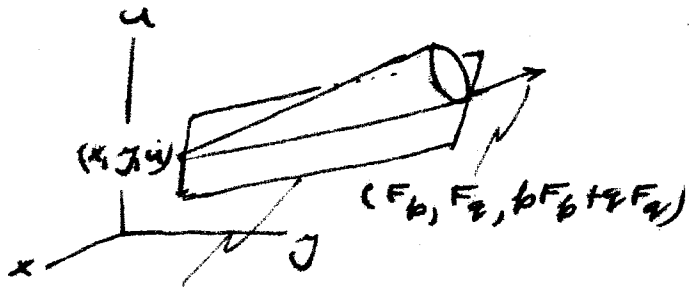
Monge cones

PDE $F(x, y, u, u_x, u_y) = 0$

Fix (x, y, u):

$F(x, y, u, p, q) = 0$

is one-param. family of planes at (x, y, u).



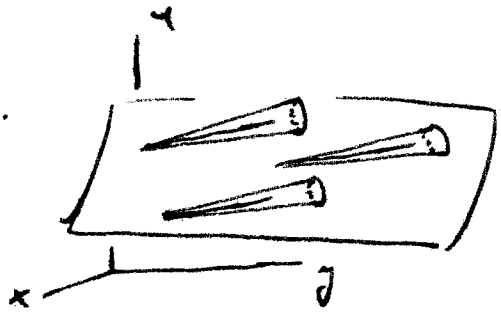
$p(x'-x) + q(y'-y) - (u'-u) = 0$

Envelope is Monge cone at (x, y, u). Obtain field of cones.

Solution: $u = \varphi(x, y)$.

Surface $u = \varphi(x, y)$ tangent to Monge cone at $(x, y, \varphi(x, y))$.

i.e. A generator of the Monge cone is tangent to the surface.



Or: Plane

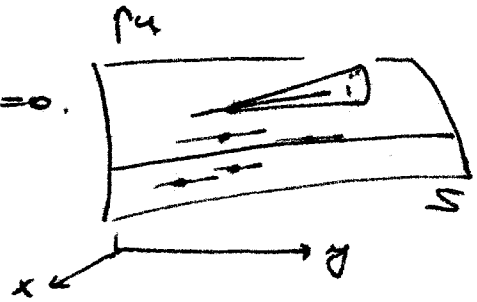
$u' - u = \varphi_x(x, y)(x'-x) + \varphi_y(x, y)(y'-y)$

is tangent Monge cone at $(x, y, \varphi(x, y))$.

Characteristic curves

$u = \varphi(x, y)$ soln of $F(x, y, u, u_x, u_y) = 0$.

At $(x, y, \varphi(x, y))$: Surface $S: u = \varphi(x, y)$ meets Monge cone in a generator of cone. Get direction field on S.



Integral curves are characteristic curves on S.

Vector field on S:

$$\left(F_p(x, y, \varphi(x, y), \varphi_x(x, y), \varphi_y(x, y)),\ F_q(\text{—}),\ \varphi_x F_p + \varphi_y F_q\right)$$

$(F_p, F_q, pF_p + qF_q)$ lies in cone. Tangent to S.

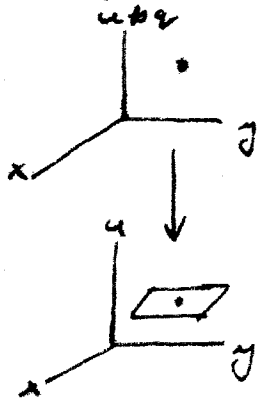
Need soln $u = \varphi(x, y)$ to write down DEs.

Charac. curve satisfies ODE!

$x = \xi(\sigma)$
$y = \eta(\sigma)$
$u = \varphi(\xi(\sigma), \eta(\sigma))$

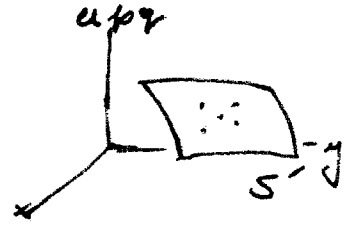
$$\frac{dx}{d\sigma} = F_p(x, y, \varphi(x, y), \varphi_x(x, y), \varphi_y(x, y))$$
$$\frac{dy}{d\sigma} = F_q(\text{———} \ '' \ \text{———})$$
$$\frac{du}{d\sigma} = \varphi_x(x, y) F_p(\text{—}) + \varphi_y(x, y) F_q(\text{—}).$$

Surface elements



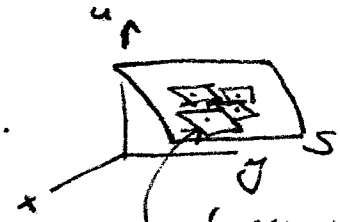
Point (x, y, u, p, q).

Plane $u' - u = p(x' - x) + q(y' - y)$.



Lifted surface S'!

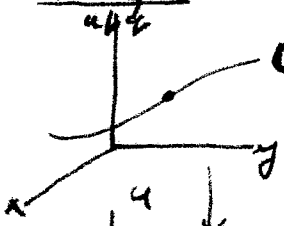
$u = \varphi(x, y)$
$p = \varphi_x(x, y)$
$q = \varphi_y(x, y)$.



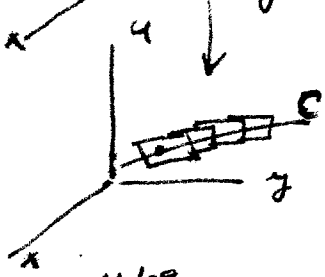
Surface S
$u = \varphi(x, y)$.

$u' - \varphi(x, y) = \varphi_x(x, y)(x' - x) + \varphi_y(x, y)(y' - y)$.

Strips



$$\begin{cases} x = x_0(t) \\ y = y_0(t) \\ u = u_0(t) \\ p = p_0(t) \\ q = q_0(t) \end{cases}$$



$$\begin{cases} x = x_0(t) \\ y = y_0(t) \\ u = u_0(t) \end{cases}$$

C' curve in $xyupq$ projects to C in xyu.

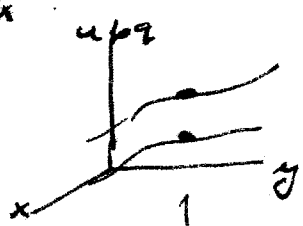
C' is a strip if basis $(xyupq)$ of C' is a plane tangent to C at (x, y, u).

i.e.

$\left(\frac{dx_0}{dt}, \frac{dy_0}{dt}, \frac{du_0}{dt}\right)$

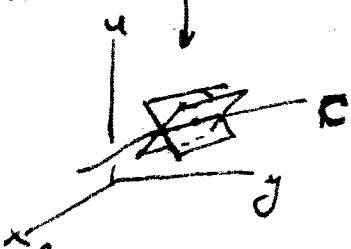
$(p_0(t), q_0(t), -1)$

Orthog. vectors.



Strip condition $\begin{cases} \frac{du_0}{dt}(t) = p_0(t)\frac{dx_0}{dt}(t) + q_0(t)\frac{dy_0}{dt}(t) \\ \text{for every } t. \end{cases}$

(OVER)



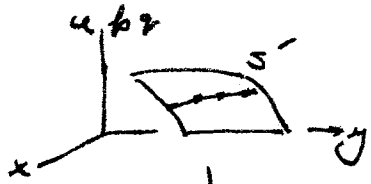
Many strips project to same C in (x, y, u).

Characteristic equations

PDE $\quad F(x, y, u, u_x, u_y) = 0$

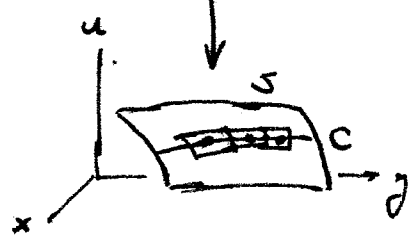
Sol'n: $\quad S: \; u = \varphi(x, y)$.

Lift S to S': $\begin{cases} u = \varphi(x, y) \\ p = \varphi_x(x, y) \\ q = \varphi_y(x, y). \end{cases}$



Characteristic on S:

$$C: \begin{cases} x = \xi(t) \\ y = \eta(t) \\ u = \varphi(\xi(t), \eta(t)) \end{cases}$$



A surface $x = x(t,\tau)$, $y = y(t,\tau)$, $z = z(t,\tau)$ need <u>not</u> be the lift of a surface in xyz. To be such it must satisfy

$$\frac{\partial z}{\partial t} = p\frac{\partial x}{\partial t} + q\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial \tau} = p\frac{\partial x}{\partial \tau} + q\frac{\partial y}{\partial \tau}$$

Lift C to a strip C' lying in S':

$$\begin{cases} x = \xi(\tau) \\ y = \eta(\tau) \\ u = \varphi(\xi(\tau), \eta(\tau)) \\ p = \varphi_x(\xi(\tau), \eta(\tau)) \\ q = \varphi_y(\xi(\tau), \eta(\tau)) \end{cases}$$

Needed below:

$$F(x, y, \varphi(x,y), \varphi_x(-), \varphi_y(-)) = 0$$

$$F_x + F_u \varphi_x + F_p \varphi_{xx} + F_q \varphi_{xy} = 0$$

$$F_y + F_u \varphi_y + F_q \varphi_{xy} + F_q \varphi_{yy} = 0.$$

Tangent direction to C':

(above)
$$\begin{cases} \dfrac{dx}{d\tau} = \dfrac{d\xi}{d\tau} = F_p(\xi, \eta, \varphi(\xi,\eta), \varphi_x(\xi,\eta), \varphi_y(\xi,\eta)) \\ \dfrac{dy}{d\tau} = \dfrac{d\eta}{d\tau} = F_q \\ \dfrac{du}{d\tau} = \varphi_x F_p + \varphi_y F_q \end{cases}$$

(above)
$$\begin{cases} \dfrac{dp}{d\tau} = \varphi_{xx}\dfrac{d\xi}{d\tau} + \varphi_{xy}\dfrac{d\eta}{d\tau} = \varphi_{xx} F_p + \varphi_{xy} F_q = -F_x - F_u \varphi_x. \quad \text{at } (\xi(\tau), \eta(\tau)) \\ \quad = -F_x(\xi, \eta, \varphi(\xi,\eta), \varphi_x(-), \varphi_y(-)) - F_u(-)\varphi_x(-). \\ \dfrac{dq}{d\tau} = -F_y - F_u \varphi_x \end{cases}$$

i.e. C' is an integral curve of the autonomous system of ODE $\longrightarrow$ CE

$$\boxed{\begin{cases} \dfrac{dx}{d\tau} = F_p(x, y, u, p, q) \\ \dfrac{dy}{d\tau} = F_q \\ \dfrac{du}{d\tau} = pF_p + qF_q \\ \dfrac{dp}{d\tau} = -F_x - pF_u \\ \dfrac{dq}{d\tau} = -F_y - qF_u \end{cases}}$$

Characteristic equations.

These eqns do not depend on S. They can be written even without knowing solution $u = \varphi(x,y)$.

<u>Characteristic strips</u>

1) Any integral curve (solution) of CE is a strip.

For! For a solution $x = x(\tau), \dots, q = q(\tau)$!

$$\underbrace{\frac{du}{d\tau} - p\frac{dx}{d\tau} - q\frac{dy}{d\tau}}_{=0 \text{ is strip condition}} = (pF_p + qF_q) - pF_p - qF_q = 0$$

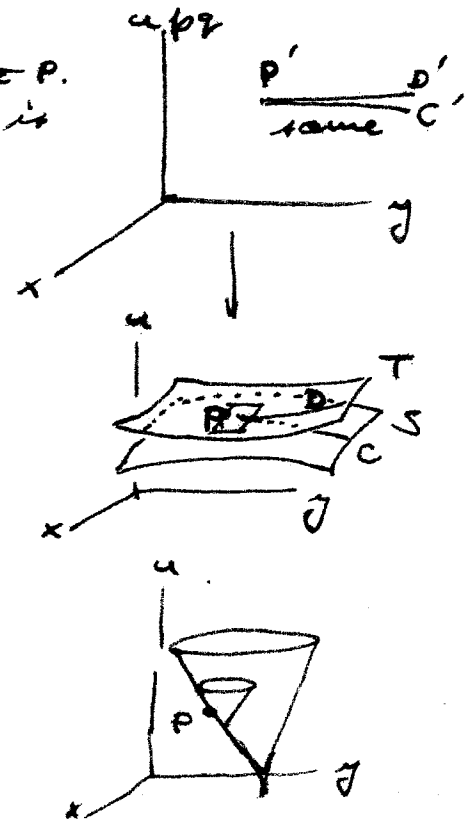
2) Any integral curve of CE for which $F=0$ at one point has $F=0$ at all points.

For! $F = F(x, y, u, p, q)$, $x = x(\tau), \dots, q = q(\tau)$!

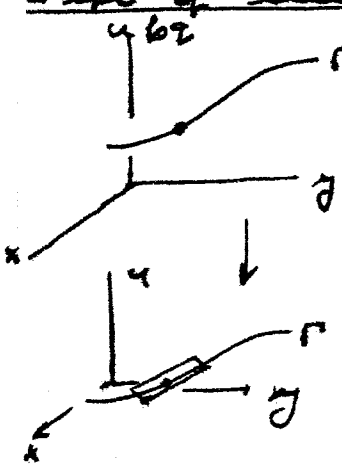
$$\frac{dF}{dt} = F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_u \frac{du}{dt} + F_p \frac{dp}{dt} + F_q \frac{dq}{dt}$$
$$= F_x F_p + F_y F_q + F_u (p F_p + q F_q)$$
$$+ F_p(-F_x - p F_u) + F_q(-F_y - q F_u) = 0 \quad \text{all } t.$$

Characteristic strip: a solution of CE lying in the hypersurface (dim $= 4$) $F(x, y, u, p, q) = 0$.
Aside: There are strips lying in $F = 0$ which are not characteristic strips.

<u>Aside</u> - on tangency of solutions of PDE $F = 0$.
S & T: two integral surfaces of $F = 0$ which are tangent at P.
P': point of (x, y, u, p, q) which is in fact common tangent plane at P.
C & D: characteristics in S & T through P (from fields on S & T from transverse curves).
C' & D': lift to S' & T'.
C' & D' are sol'ns of ~~same~~ ~~CE~~ CE through <u>same</u> point P'. So $C' = D'$.
i.e. S & T are tangent along a curve which is a characteristic curve for each surface.



<u>Lift of initial data</u> $F(x, y, u, p, q)$ class C^2.



Lift to a strip lying in $F = 0$:

$$\Gamma' \begin{cases} x = x_0(t) \\ y = y_0(t) \\ u = u_0(t) \\ p = p_0(t) \\ q = q_0(t) \end{cases}$$

Find suitable $p_0(t)$, $q_0(t)$. Class C^1.

Given initial curve:

$$\Gamma \begin{cases} x = x_0(t) \\ y = y_0(t) \\ u = u_0(t) \end{cases}$$

Find $x_0(t)$, $y_0(t)$, $u_0(t)$ of class C^2.

i.e. Find $p_0(t)$, $q_0(t)$ so that:

$$\begin{cases} \frac{du_0}{dt}(t) = p_0(t)\frac{dx_0}{dt}(t) + q_0(t)\frac{dy_0}{dt}(t) & \text{strip condition} \\ F(x_0(t), y_0(t), u_0(t), p_0(t), q_0(t)) = 0 & \text{lies in } F=0 \end{cases}$$

i.e. Solve for $p = p_0(t)$, $q = q_0(t)$: class C^1.

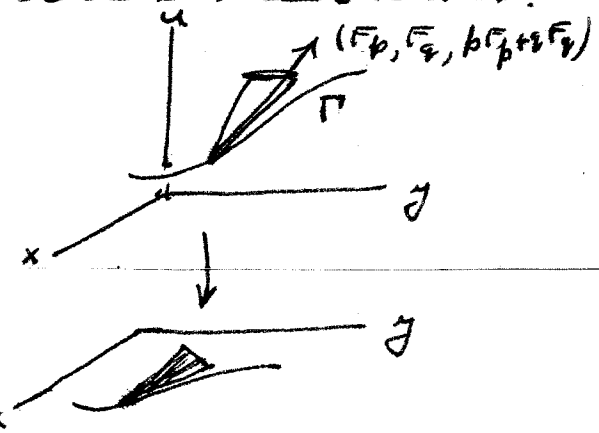
$$\begin{cases} f = p\frac{dx_0}{dt}(t) + q\frac{dy_0}{dt}(t) - \frac{du_0}{dt}(t) \\ F(x_0(t), y_0(t), u_0(t), p, q) = 0 \end{cases}$$

$$\frac{\partial(f,F)}{\partial(p,q)} = \frac{dx_0}{dt}F_q - \frac{dy_0}{dt}F_p = \Delta(t,p,q)$$

$$\Delta(t,p,q) = \frac{dx_0}{dt}(t)F_q(x_0(t), y_0(t), u_0(t), p, q) - \frac{dy_0}{dt}(t)F_p(\text{—}, \text{—}, \text{—}, p, q)$$

If $\Delta(t,p,q) \neq 0$, can solve for p & q.

If $\Delta \neq 0$ at one value of t, it lives near this t.



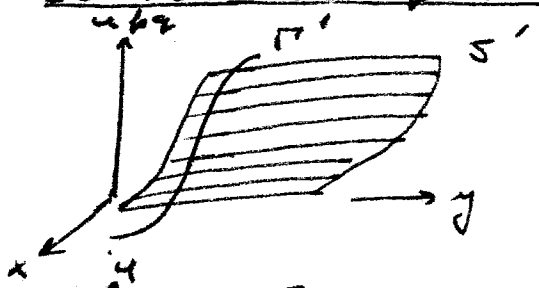
$\Delta(t,p,q) \neq 0$ is:

$$\begin{cases} \left(\frac{dx_0}{dt}(t), \frac{dy_0}{dt}(t)\right) \\ \left(F_p(x_0(t), y_0(t), u_0(t), p, q), F_q(\text{—})\right) \end{cases}$$

not parallel.

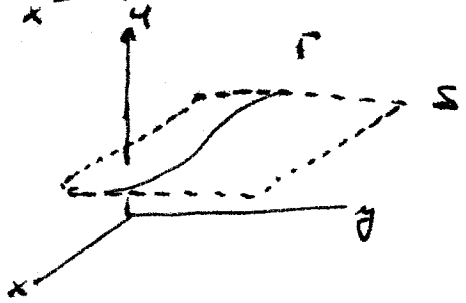
i.e. At $(x_0(t), y_0(t), u_0(t))$, choose p & q so that plane $u' - u_0 = p(x' - x_0) + q(y' - y_0)$ meets range cone in a line having different projection on xy-plane than the tangent to Γ.

<u>Construction of solution</u> F and Γ class C^2.



Γ': Γ lifted to a strip in $F=0$.

S': surface formed by ~~forming~~ characteristic strips passing through points of Γ'.



Then: S' is the lift of a surface S which is integral surface of PDE. $F(x, y, u, u_x, u_y) = 0$.

<u>DE</u> is CE. Class C^1. <u>EC</u> is Γ'. Class C^1.

$$\begin{cases} \frac{dx}{dt} = F_p(x,y,u,p,q) \\ \frac{dy}{dt} = F_q \\ \frac{du}{dt} = pF_p + qF_q \\ \frac{dp}{dt} = -F_x - pF_u \\ \frac{dq}{dt} = -F_y - qF_u \end{cases} \qquad \begin{cases} x = x_0(t) \\ y = y_0(t) \\ u = u_0(t) \\ p = p_0(t) \\ q = q_0(t) \end{cases} \text{ when } \tau = 0.$$

<u>Soln</u> - class C^1.

$$S' \begin{cases} x = X(t,\tau) \\ y = Y(t,\tau) \\ u = U(t,\tau) \\ p = P(t,\tau) \\ q = Q(t,\tau) \end{cases} \quad \left.\begin{matrix} x = X(t,\tau) \\ y = Y(t,\tau) \\ u = U(t,\tau) \end{matrix}\right\} S \qquad \begin{cases} X(t,0) = x_0(t) \\ Y(t,0) = y_0(t) \\ U(t,0) = u_0(t) \\ P(t,0) = p_0(t) \\ Q(t,0) = q_0(t). \end{cases}$$

Parametric surface.

<u>Surface S</u> in non-parametric form $u = \varphi(x,y)$.

$$\begin{cases} x = X(t,\tau) \\ y = Y(t,\tau) \end{cases} \to \begin{cases} t = S(x,y) \\ \tau = T(x,y) \end{cases} \to u = \varphi(x,y) = U(S(x,y), T(x,y)).$$

$\frac{\partial(X,Y)}{\partial(t,\tau)}(t,\tau) = X_t Y_\tau - Y_t X_\tau$ <u>not</u> 0 for τ small by:

$$\frac{\partial(X,Y)}{\partial(t,\tau)}(t,0) = \frac{dx_0}{dt}(t)\, F_q(x_0(t), y_0(t), u_0(t), p_0(t), q_0(t))$$
$$- \frac{dy_0}{dt}(t)\, F_p(x_0(t), -, -, -, -)$$
$$= \Delta(t, p_0(t), q_0(t)) \neq 0.$$

<u>Claim 1</u>: S' lies in hypersurface $F(x,y,u,p,q) = 0$.
For: Each characteristic strip start at a point of Γ' which lies in $F=0$, so ~~strip~~ char. strip stays in $F=0$.

<u>Claim 2</u>: S' is the lift $\begin{cases} u = \varphi(x,y) \\ p = \varphi_x(x,y) \\ q = \varphi_y(x,y) \end{cases}$ of S.

i.e. $\begin{cases} P(t,\tau) = \varphi_x(X(t,\tau), Y(t,\tau)) \\ Q(t,\tau) = \varphi_y(\text{——}, \text{——}) \end{cases}$ <u>See below</u>.

<u>Consequence</u>. $u = \varphi(x,y)$ is solution of $F(x,y,u,u_x,u_y) = 0$ and passes through Γ.
Note: φ is class C^1.

<u>Proof of Claim 2.</u>

1) $A(t,\tau) = U_t - PX_t - QY_t \longrightarrow PX_t + QY_t = U_t$

$B(t,\tau) = U_\tau - PX_\tau - QY_\tau \qquad \varphi_x X_t + \varphi_y Y_t = U_t$

Shown below $A \equiv 0$ & $B \equiv 0$. $\qquad PX_\tau + QY_\tau = U_\tau$

$U(t,\tau) = \varphi(X(t,\tau), Y(t,\tau)) \longrightarrow \varphi_x X_\tau + \varphi_y Y_\tau = U_\tau$

$$\begin{cases} (P-\varphi_x)X_t + (Q-\varphi_y)Y_t = 0 \\ (P-\varphi_x)X_\tau + (Q-\varphi_y)Y_\tau = 0 \end{cases} \xrightarrow[\Delta \neq 0]{} \begin{array}{l} P = \varphi_x \\ Q = \varphi_y \end{array}.$$

<u>Or</u>: $\varphi(x,y) = U(S(x,y), T(x,y))$

$$\varphi_x = U_t S_x + U_\tau T_x = (PX_t + QY_t)S_x + (PX_\tau + QY_\tau)T_x$$
$$= P(X_t S_x + X_\tau T_x) + Q(Y_t S_x + Y_\tau T_x)$$
$$= P \frac{\partial}{\partial x} \underbrace{X(S(x,y), T(x,y))}_{x} + Q \frac{\partial}{\partial y} \underbrace{Y(-,-)}_{y} = P$$

$\varphi_y = Q$ likewise.

2) $A(t,\tau) = U_t - PX_t - QY_t$ | $X(t,\tau), \dots, Q(t,\tau)$ solns of CE.

$B(t,\tau) = U_\tau - PX_\tau - QY_\tau$

$$\frac{\partial A}{\partial \tau} = U_{t\tau} - P_\tau X_t - PX_{t\tau} - Q_\tau Y_t - QY_{t\tau}$$

$$\frac{\partial B}{\partial t} = U_{t\tau} - P_t X_\tau - PX_{t\tau} - Q_t Y_\tau - QY_{t\tau}$$

$$\frac{\partial A}{\partial \tau} - \frac{\partial B}{\partial t} = -P_\tau X_t + P_t X_\tau - Q_\tau Y_t + Q_t Y_\tau$$

$$= -(-F_x - PF_u)X_t + P_t F_p - (-F_y - QF_u)Y_t + Q_t F_q$$
$$= F_x X_t + F_y Y_t + F_u(PX_t + QY_t) + F_p P_t + F_q Q_t$$

$G(t,\tau) = F(X(t,\tau), Y(t,\tau), U(t,\tau), P(t,\tau), Q(t,\tau))$.

$$\frac{\partial G}{\partial t} = F_x X_t + F_y Y_t + F_u U_t + F_p P_t + F_q Q_t$$

$$\frac{\partial A}{\partial \tau} - \frac{\partial B}{\partial t} - \frac{\partial G}{\partial t} = F_u(PX_t + QY_t - U_t) = -F_u A.$$

3) $B \equiv 0$ since every soln of CE is a strip.

$G \equiv 0$ since S' lies in $F = 0$.

$\frac{\partial A}{\partial \tau} = -F_u A$ ODE in τ, t fixed.

$$A(t,\tau) = A(t,0)\exp\left(-\int_0^\tau F_u(X(t,\tau),-,-)\,d\tau\right).$$

$$A(t,0) = U_t(t,0) - P(t,0)X_t(t,0) - Q(t,0)Y_t(t,0)$$
$$= \frac{du_0}{dt}(t) - p_0(t)\frac{dx_0}{dt}(t) - q_0(t)\frac{dy_0}{dt}(t) = 0$$

$\Rightarrow A \equiv 0.$

Quasi-linear equations.

PDE $\quad A(x,y,u)\,u_x + B(x,y,u)\,u_y = C(x,y,u)$

Planar $\quad F(x,y,u,p,q) = Ap + Bq - C.$

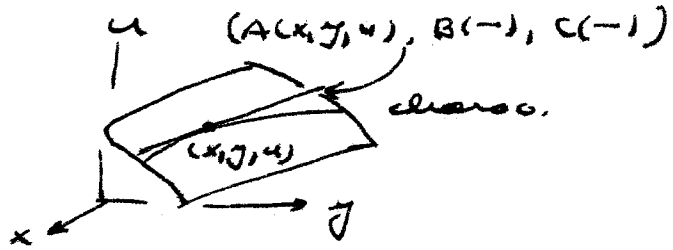
Monge cone.

$F_p = A$

$F_q = B$ $\qquad F=0$

$pF_p + qF_q = Ap + Bq = C$

Monge cone degenerates to a line.



Characteristic equations

$$\text{CE}\begin{cases} \dfrac{dx}{dt} = A \\ \dfrac{dy}{dt} = B \\ \dfrac{du}{dt} = Ap + Bq \\ \dfrac{dp}{dt} = -(A_x p + B_x q - C_x) - p(A_u p + B_u q - C_u) \\ \dfrac{dq}{dt} = -(-) - q(-) \end{cases}$$

$$\boxed{\begin{aligned} \frac{dx}{dt} &= A(x,y,u) \\ \frac{dy}{dt} &= B(x,y,u) \\ \frac{du}{dt} &= C(x,y,u) \end{aligned}}$$

Characteristic equations.

A, B, C of class C^1 suffices.

If a solution starts in $F=0$, it stays in $F=0$. Hence $u = U(s,t)$ satisfies $\dfrac{du}{dt} = Ap + Bq = C$.

i.e. X, Y, U can be determined by solving local char. eqns.

$$\begin{cases} x = X(s,t) \\ y = Y(s,t) \\ u = U(s,t) \\ p = P(s,t) \\ q = Q(s,t) \end{cases}$$

Initial data. Need $\Delta(s, p_0(s), q_0(s)) \neq 0$.

$$\Delta(s,p,q) = \frac{dx_0}{ds} F_q - \frac{dy_0}{ds} F_p = \frac{dx_0}{ds}(s)\, B(x_0(s), y_0(s), u_0(s)) - \frac{dy_0}{ds}(s)\, A(x_0(s), y_0(s), u_0(s))$$

Independent of p & q.

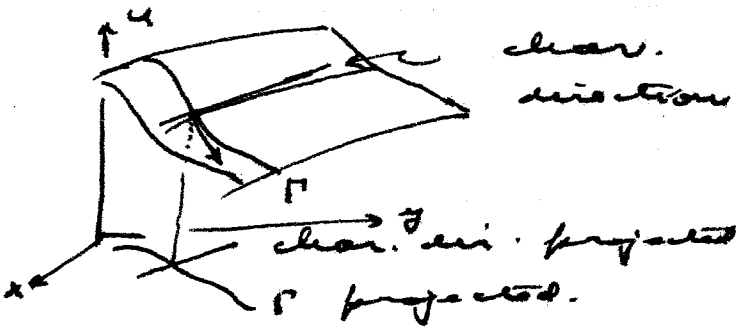
i.e. $\begin{cases} p\dfrac{dx_0}{dt} + q\dfrac{dy_0}{dt} = \dfrac{du_0}{dt} \\ Ap + Bq = C \end{cases}$

Unique soln $p = p_0(s)$, $q = q_0(s)$ for $\Delta \neq 0$.

Γ lifts uniquely to a strip Γ' in $F=0$.

Note $\Delta \neq 0$ is needed later also to change from s,t to x,y as parameters of soln surface S.

i.e. $\Delta(s) \neq 0$ means: Tangent to Γ and charac. direction at $(x_0(s), y_0(s), u_0(s))$, when projected to xy plane, are <u>not</u> parallel.



<u>Special case</u> $\quad A(x,y)\,u_x + B(x,y)\,u_y = C(x,y,u)$

$$CE\begin{cases} \dfrac{dx}{dt} = A(x,y) \\ \dfrac{dy}{dt} = B(x,y) \\ \dfrac{du}{dt} = C(x,y,u) \end{cases}$$

Solve for $x = X(s,t)$, $y = Y(s,t)$.
Solve for $u = U(s,t)$ from $\dfrac{du}{dt} = C(X(s,t), Y(s,t), u)$.

Note: Projection of charac. in xy-plane satisfy: $\quad B(x,y)\,dx - A(x,y)\,dy = 0.$

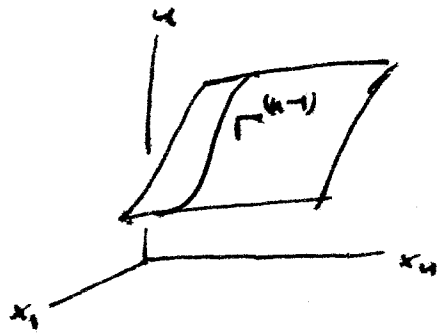
Several independent variables

PDE $\quad F\left(x_1, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) = 0 \quad , \quad u_{x_1} = \frac{\partial u}{\partial x_1}$ etc.

Solns $\quad u = \varphi(x_1, \dots, x_n)$.

Initial data

Γ is $(n-1)$-dim'l
$$\begin{cases} x_1 = x_{01}(t_1, \dots, t_{n-1}) \\ x_2 = x_{02}(t_1, \dots, t_{n-1}) \\ \vdots \\ x_n = x_{0n}(t_1, \dots, t_{n-1}) \\ u = u_0(t_1, \dots, t_{n-1}) \end{cases}$$



Many cons $\quad$ Generate:

Prime a new curve
$$\begin{cases} x_1' = x_1 + \tau F_{p_1}(x_1, \dots, x_n, u, p_1, \dots, p_n) \\ \vdots \\ x_n' = x_n + \tau F_{p_n}(\qquad\qquad\qquad) \\ u' = u + \tau(p_1 F_{p_1} + p_2 F_{p_2} + \dots + p_n F_{p_n}). \end{cases}$$

$$F(x_1, \dots, x_n, u, p_1, \dots, p_n) = 0$$

Characteristic equations

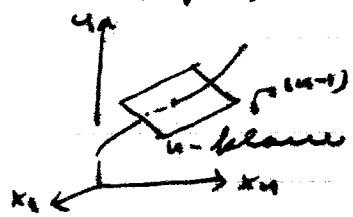
$$\text{CE} \begin{cases} \frac{dx_1}{d\tau} = F_{p_1} \\ \vdots \\ \frac{dx_n}{d\tau} = F_{p_n} \\ \frac{du}{d\tau} = p_1 F_{p_1} + \dots + p_n F_{p_n} \\ \frac{dp_1}{d\tau} = -F_{x_1} - p_1 F_u \\ \vdots \\ \frac{dp_n}{d\tau} = -F_{x_n} - p_n F_u \end{cases}$$

Abbreviate as:
$$\begin{cases} \frac{dx}{d\tau} = F_p \\ \frac{du}{d\tau} = p F_p \\ \frac{dp}{d\tau} = -F_x - p F_u . \end{cases}$$

Strip condition

$$\begin{cases} x_1 = x_{01}(t_1, \dots, t_{n-1}) \\ \vdots \\ x_n = x_{0n}(t_1, \dots, t_{n-1}) \\ u = u_0(t_1, \dots, t_{n-1}) \\ p_1 = p_{01}(t_1, \dots, t_{n-1}) \\ \vdots \\ p_n = p_{0n}(t_1, \dots, t_{n-1}) \end{cases}$$

$$\begin{cases} \frac{\partial u_0}{\partial t_i} = \sum_{r=1}^{n} p_{0r} \frac{\partial x_{0r}}{\partial t_i} \\ i = 1, 2, \dots, n-1 \end{cases}$$



Lift of initial data

$$\begin{cases} \dfrac{\partial u_0}{\partial t_1} = p_1 \dfrac{\partial x_{01}}{\partial t_1} + \cdots + p_n \dfrac{\partial x_{0n}}{\partial t_1} \\ \vdots \\ \dfrac{\partial u_0}{\partial t_{n-1}} = p_1 \dfrac{\partial x_{01}}{\partial t_{n-1}} + \cdots + p_n \dfrac{\partial x_{0n}}{\partial t_{n-1}} \\ F(\underbrace{x_{01}, \cdots, x_{0n}, u_0}_{\text{Known fcns of } t_1, \cdots, t_{n-1}}, \underbrace{p_1, \cdots, p_n}_{\text{Unknowns.}}) = 0 \end{cases}$$

Solve for:

$$\begin{cases} p_1 = p_{01}(t_1, \cdots, t_{n-1}) \\ \vdots \\ p_n = p_{0n}(\text{——}). \end{cases}$$

Jacobian condition

$$\Delta(t_1, \cdots, t_{n-1}, p_1, \cdots, p_n) = \begin{vmatrix} \dfrac{\partial x_{01}}{\partial t_1} & \cdots & \dfrac{\partial x_{0n}}{\partial t_1} \\ \vdots & & \vdots \\ \dfrac{\partial x_{01}}{\partial t_{n-1}} & \cdots & \dfrac{\partial x_{0n}}{\partial t_{n-1}} \\ F_{p_1} & \cdots & F_{p_n} \end{vmatrix}$$

Solution of CE EPDE

$$\begin{cases} x_1 = X_1(t_1, \cdots, t_{n-1}, \tau) \\ \vdots \\ x_n = X_n \\ u = U \\ p_1 = P_1 \\ \vdots \\ p_n = P_n \end{cases}$$

Solve for $t_1, \cdots, t_{n-1}, \tau$ as fcns of $x_1, \cdots, x_n$ & substitute into $u = U$.

$X_1(t_1, \cdots, t_{n-1}, 0) = x_{01}(t_1, \cdots, t_{n-1})$, etc.