Method of characteristics - Courant

**Range cone**

**PDE:** $F(x, y, u, u_x, u_y) = 0$

- **Fix** $(x, y, u)$:
  - $F(x, y, u, u_x, u_y) = 0$
  - is a surface.

  A family of planes at $(x, y, u)$:
  - Envelopes in range cone at $(x, y, u)$.
  - Obtain field of cones.

**Solutions:** $u = \phi(x, y)$.

- Surface $u = \phi(x, y)$ tangent to range cone at $(x, y, \phi(x, y))$.
  - A generation of the range cone is tangent to the surface.

Or: Plane

$$u = \phi(x, y) (y'-y) + \phi(x, y) (x'-x)$$

is tangent range cone at $(x, y, \phi(x, y))$.

**Characteristic curves**

$u = \phi(x, y)$, solve of $F(x, y, u, u_x, u_y) = 0$.

At $(x, y, \phi(x, y))$:

- Surface $S$: $u = \phi(x, y)$
  - meets range cone in a generation of cone.
  - Get direction field on $S$.

**Tangential curves** are characteristic curves on $S$.

Vector field on $S$:

$$\left( \begin{array}{c} F_x(x, y, \phi(x, y), u(x, y, u_x, u_y)) \\ F_y(x, y, \phi(x, y), u(x, y, u_x, u_y)) \end{array} \right)$$

- $F_x, F_y, F_x+2F_y$ lies in cone: Tangent to $S$.

**Characteristic curve** satisfies ODE! The curve traced $u \equiv \phi$.
Surface elements

\[ \begin{align*}
& u = \psi(x,y) \\
& v = \phi_x(x,y) \\
& w = \phi_y(x,y)
\end{align*} \]

Surface \( S \)

\[ u = \phi(x,y) \]

Lifted surface \( S' \)

\[ \begin{align*}
& u' = \psi(x,y) \\
& v' = \phi_x(x,y) \\
& w' = \phi_y(x,y)
\end{align*} \]

Plane

\[ u' - u = b(x-1) + c(y-y_0) \]

\[ \begin{align*}
& u''(x,y) = \phi_x(x,y)(x-1) \\
& u''(x,y)(y' - y_0)
\end{align*} \]

\[ \begin{align*}
& \frac{dx}{dt} = \phi_x(x,y) \\
& \frac{dy}{dt} = \phi_y(x,y) \\
& \frac{dz}{dt} = \psi(x,y)
\end{align*} \]

\[ \begin{align*}
& \frac{dx}{dt} = \phi_x(x,y) + \phi_y(x,y) \\
& \frac{dy}{dt} = \phi_x(x,y) + \phi_y(x,y)
\end{align*} \]

Many strips project to same \( C \) in \((x,y)\).
A surface \( x = x(u,v), \ y = y(u,v), \ z = z(u,v) \) need not be the lift of a surface in \( xy \). To be such, it must satisfy

\[
\frac{\partial x}{\partial u} = b \frac{\partial x}{\partial u} + a \frac{\partial y}{\partial u} \\
\frac{\partial x}{\partial v} = b \frac{\partial x}{\partial v} + a \frac{\partial y}{\partial v}
\]
Lift $C$ to a strip $C'$

$$\begin{align*}
x &= 3 (t) \\
y &= 4 (t) \\
z &= q(3(t), 4(t)) \\
\phi &= q_x (3(t), 4(t)) \\
f &= q_y (3(t), 4(t))
\end{align*}$$

Tangent direction to $C'$:

$$\begin{align*}
\frac{dx}{dt} &= \frac{d3}{dt} = F_p (3, 4, q(3, 4), q_x (3, 4), q_y (3, 4)) \\
\frac{dy}{dt} &= \frac{d4}{dt} = F_q \\
\frac{dz}{dt} &= q_x F_p + q_y F_q \\
\frac{d\phi}{dt} &= q_{xx} F_p + q_{xy} F_q - F_x - F_u q_x \\
\frac{df}{dt} &= -F_y - F_u q_y
\end{align*}$$

$C'$ is an integral curve of the autonomous system of ODEs:

$$\begin{align*}
\frac{dx}{dt} &= F_p (x, y, q_x, q_y) \\
\frac{dy}{dt} &= F_q \\
\frac{dz}{dt} &= F_u + \phi F_q \\
\frac{d\phi}{dt} &= F_x - \phi F_u \\
\frac{df}{dt} &= -F_y - \phi F_u
\end{align*}$$

Characteristic equations:

1) Any integral curve (solution) of $CE$ is a strip.

For $x = x(t), \cdots, z = z(t)$

$$\begin{align*}
\frac{d\phi}{dt} - \phi \frac{dx}{dt} - \phi \frac{dz}{dt} &= (\phi F_p + \phi F_q) - \phi F_p - \phi F_q = 0 \\
= 0 & \text{ is strip condition}
\end{align*}$$

2) Any integral curve of $CE$ for which $F = 0$ at one point has $F = 0$ at all points.

For $F = F(x, y, z, q, \phi, \psi)$, $x = x(t), \cdots, z = z(t)$;
\[ \frac{dF}{dt} = F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt} + F_\theta \frac{d\theta}{dt} + F_\phi \frac{d\phi}{dt} \]

\[ = F_x \frac{d\phi}{dt} + F_y \frac{d\phi}{dt} + F_z \left( kF_\phi + 2F_\phi \right) \\
+ F_\phi \left( -F_x - kF_\phi \right) + F_\phi \left( -F_y - 2F_\phi \right) = 0 \quad \text{for } \phi \]

Characteristic surface: a set of CE lying on the hypersurface \( \text{and } \sigma = \text{as} \) \( F(x, y, u, v, \theta) = 0 \).

Aside 1: These surface lying in \( F = 0 \) variables are not characteristic surfaces.

Aside 2: On tangents of solutions of PDE \( F = 0 \).

Set T: lines integral surfaces of \( F = 0 \) which are tangent at \( \mathbf{P} \).

\( \mathbf{P}' \): point \( (x, y, u, v, \theta) \) which is a plane common tangent plane at \( \mathbf{P} \).

C \& D: characteristics in \( \text{set } T \) through \( \mathbf{P} ' \) (from fields on \( \text{set } T \) from range curve).

C \& D: lift to \( \text{set } T ' \).

C \& D' are lines of \( \text{CE through } \mathbf{P}' \).

\( \text{CE through } \mathbf{P}' \) point \( \mathbf{P}' \). So \( C' = D' \)

\( \therefore \) \( \text{set } T \) are tangent along a curve which is a characteristic curve for each surface.

Lift of initial data \( F(x_0, y_0, u_0, v_0, \theta_0) \) class \( C^2 \):

\[ \begin{align*}
\mathbf{P}' & \quad \text{such that } \mathbf{P}' \in T' \\
\text{Fluid points} & \\
\mathbf{F} & \quad \text{from } x_0(\tau), y_0(\tau), u_0(\tau) \\
\text{Governing initial curve!} & \quad \text{of class } C^1
\end{align*} \]
\[ F(x, y, u, v) = 0 \]

**Problem:**

\[ \begin{aligned}
    \frac{dx}{dt} &= u(+) + z_0(+) \frac{du}{dt}(+) \\
    \frac{dy}{dt} &= v(+) + z_0(+) \frac{dv}{dt}(+) \\
    F(x, y, u, v) &= 0
\end{aligned} \]

Let \( \mathbf{F} = 0 \) and \( \mathbf{F} = 0 \).

**Solution:**

\[ \begin{aligned}
    f &= f_0(+) + g_0(+) - \frac{dx_0}{dt}(+) \\
    F(x, y, u, v) &= 0
\end{aligned} \]

\[ \frac{\partial (f_0, F)}{\partial (x_0, y)} = \frac{dx_0}{dt} F_x - \frac{dy_0}{dt} F_y = \Delta (+, k, y) \]

\[ \Delta (+, k, y) = \frac{dx_0}{dt} F_x(x_0(+), y_0(+), u_0(+), v_0(+), k, y) - \frac{dy_0}{dt} F_y(x_0(+), y_0(+), u_0(+), v_0(+), k, y) \]

If \( \Delta (+, k, y) \neq 0 \), we can solve for \( k \in \mathbb{R} \).

If \( \Delta \to 0 \) at one value of \( + \), it has no regular point.

\[ \Delta (+, k, y) = 0 \quad \text{at:} \]

\[ \left\{ \left( \frac{dx_0}{dt}(+), \frac{dy_0}{dt}(+) \right) \right\} \]

\[ \left\{ (F_p(x_0(+), y_0(+), u_0(+), v_0(+), k, y), F_{x0}(+)) \right\} \]

These are not parallel.

Choose \( k \in \mathbb{R} \) so that plane \( x' = x_0 = k(x' - x_0) + z(y' - y_0) \) meets \( x \)-plane in a curve having different projections on \( xy \)-plane than the tangent to \( \Gamma \).

**Construction of solution**

\[ F \] and \( \Gamma \) class \( C^2 \).

\[ \Gamma' : \Gamma \text{ lifted to a strip in } F = 0. \]

\[ S' : \text{surface formed by characteristic} \]

\[ \text{touching characteristic} \]

\[ \text{family of } \Gamma'. \]

Then: \( S' \) is the lift of a surface \( S \) which is \( C^1 \) integral surface of \( F \).

\[ F(x, y, u, v, u_x, v_y) = 0. \]
\text{Case C. Class } C^1.
\begin{align*}
\frac{dx}{dt} &= F_p(x, y, u, b, \gamma) \\
\frac{dy}{dt} &= F_q \\
\frac{du}{dt} &= b F_p + \theta F_q \\
\frac{db}{dt} &= -F_x - b F_y \\
\frac{d\gamma}{dt} &= -F_y - b F_x
\end{align*}

\text{Case C. Class } C^1.
\begin{align*}
x &= x_0(t) \\
y &= y_0(t) \\
u &= u_0(t) \\
b &= b_0(t) \\
\gamma &= \gamma_0(t)
\end{align*}

\text{Surface } S \text{ in non-parametric form } u = \varphi(x, y).
\begin{align*}
x = X(t, \xi) \\
y = Y(t, \xi) \\
u = U(t, \xi) \\
b = P(t, \xi) \\
\gamma = Q(t, \xi)
\end{align*}

\text{Parameteric surface.}
\begin{align*}
X(t, \xi) = x_0(t) \\
Y(t, \xi) = y_0(t) \\
U(t, \xi) = u_0(t) \\
P(t, \xi) = p_0(t) \\
Q(t, \xi) = q_0(t)
\end{align*}

\text{Claim: } S' \text{ lies in hypersurface } F(x, y, u, b, \gamma) = 0.
\text{For! Each characteristic \textit{still} starts at a point of } S \text{, which lives in } F = 0, \text{ so \textit{still} claim.}
\text{Stays in } F = 0.

\text{Claim: } S' \text{ is still lift of } S. \text{ \textit{See below.}}
\begin{align*}
u &= \varphi_x(X(t, \xi), Y(t, \xi)) \\
b &= \varphi_y(X(t, \xi), Y(t, \xi)) \\
\gamma &= \varphi_\gamma(X(t, \xi), Y(t, \xi))
\end{align*}

\text{Consequence. } u = \varphi(x, y) \text{ is a solution of } F(x, y, u, b, \gamma) = 0 \text{ and passes through } \Gamma.
\text{Note: } \varphi \text{ is class } C^1.
Proof of Claim 2.

1) \( A(x, t) = U_t - P K_x - Q Y_t \to PX_x + Q Y_t = U_t \)
   \( B(x, t) = U_t - P K_x - Q Y_t \to Q_x K_x + P Y_t = U_t \)

Show Claim \( A \equiv 0 \& B \equiv 0 \).

\[ p X_t + Q Y_t = U_t \]
\[ U(x, t) = \varphi (x, t, y, u_x, y_t) \to qx K_x + py y_t = U_t \]

\[ \begin{cases} 
(P - P_x) X_t + (Q - P_y) Y_t = 0 \quad \rightarrow \quad P = q_x \\
(P - P_x) X_t + (Q - P_y) Y_t = 0 \quad A \equiv 0 \quad Q = q_y 
\end{cases} \]

\[ \varphi (x, y, y') = U(X(x, y', T_{x+y}), y', y') \]

\[ \varphi_x = U_x + U_t X_t = (P X_x + Q Y_t) X_t + (P K_x + Q Y_t) T_x \]

\[ = P (X_x S_x + X_t T_x) + Q (K_x S_x + Y_t T_x) \]

\[ = P \frac{\partial}{\partial t} \chi (S(x, y', T_{x+y}), y', y') + Q \frac{\partial y}{\partial t} \gamma (-, -) = P \]

\[ q_y = q \quad \text{LaCorma} \]

2) \( A(x, t) = U_t - P K_x - Q Y_t \)
   \( B(x, t) = U_t - P K_x - Q Y_t \)

\[ \frac{\partial A}{\partial x} = U_t - P K_x - P K_x - Q Y_t + Q Y_t \]
\[ \frac{\partial B}{\partial x} = U_t - P K_x - P K_x - Q Y_t + Q Y_t \]

\[ \frac{\partial A}{\partial t} - \frac{\partial B}{\partial x} = -P K_x + P K_x - Q Y_t + Q Y_t \]

\[ = -(P F_x - P F_u) X_t + P F_t - (-P K_x + Q Y_t) Y_t + Q Y_t \]

\[ \frac{\partial A}{\partial t} = F_x X_t + F_0 Y_t + F_u U_t + F_t P + F_p P_Y + F_y Q \]

\[ G(x, t) = F(X(x, t), Y(t, x), U(t, x), P(t, x), Q(t, x)) \]

\[ \frac{\partial G}{\partial x} = F_x X_t + F_0 Y_t + F_u U_t + F_t P + F_p P_Y + F_y Q \]

\[ \frac{\partial A}{\partial x} - \frac{\partial B}{\partial x} = F_u (P K_x + Q Y_t - U_t) = -F_u A. \]

3) \( B \equiv 0 \quad \text{since every member of } CE \to a \text{ Strict}. \)

\( G \equiv 0 \quad \text{since } S', \text{ and } F = 0. \)

\[ \frac{\partial A}{\partial x} = -F_u A \quad \text{since } CE \to a \text{ Strict}. \]

\[ A(x, t) = \int_{-1}^{x} F_u (X(x, t, t)) \, dx \]

\[ A(x, t) = U(x, t) - P(x, t) X_t - Q(x, t) Y_t \]

\[ = \frac{d}{dt} (-) - F_u (\cdot) \frac{d}{dt} (\cdot) - \mathcal{L} (\cdot) \frac{d}{dt} (\cdot) \equiv 0 \]

\[ \Rightarrow A \equiv 0. \]
Quasi-linear equations.

\[ PDE \quad A(x, y, u) \frac{dx}{dt} + B(x, y, u) \frac{dy}{dt} = C(x, y, u) \]

Plane \[ F(x, y, u, \rho, \tau) = \rho \hat{B} + B \tau = C \]

Characteristic equation:

\[ \begin{align*}
& \frac{dx}{dt} = A \\
& \frac{dy}{dt} = B \\
& \frac{dz}{dt} = A \hat{B} + B \tau \\
& \frac{d\rho}{dt} = -(A \hat{B} + B \tau - C) \\
& \frac{d\tau}{dt} = -\Lambda(A \hat{B} + B \tau - C) \\
& \frac{du}{dt} = -(\Lambda - \Lambda \Lambda(-))
\end{align*} \]

If \( A, B, C \) of class \( C^1 \), sufficient.

Characteristic equation:

\[ \begin{align*}
& \frac{dx}{dt} = A(x, y, u) \\
& \frac{dy}{dt} = B(x, y, u) \\
& \frac{dz}{dt} = C(x, y, u)
\end{align*} \]

\( \begin{align*}
& x = X(t, s) \\
& y = Y(t, s) \\
& u = U(t, s) \\
& \rho = P(t, s) \\
& \tau = Q(t, s)
\end{align*} \]

Initial data: Need \( \Delta(x_0, y_0, z_0) \neq 0 \).

\[ \Delta(x, y, z) = \frac{d}{dt} \frac{F(x, y, z, \rho, \tau)}{\rho} - \frac{d}{dt} \frac{F(x, y, z, \rho, \tau)}{\tau} = \frac{d}{dt} (x) B(x, y, z, \rho, \tau, \Delta(x, y, z, \rho, \tau)) \]

In particular if \( k \in \mathbb{R} \),

\[ \begin{align*}
& k \frac{d}{dt} x + \tau \frac{d}{dt} z = \frac{d}{dt} (x) B(x, y, z, \rho, \tau) \\
& A \hat{B} + B \tau = C
\end{align*} \]

A unique solution

\[ \begin{align*}
& b = k d(x), \tau = z d(x) \\
& \Delta \neq 0.
\end{align*} \]

\( \Gamma \) lies uniquely on a surface \( F' \) in \( F = 0 \).

Note: \( \Delta \neq 0 \) is needed after change from \( x, y, z \) to \( r, s, \tau \) as parameters of surface \( S \).
\[ A(x, y) = B(x, y) \]

\[ S \text{ is the set } \{ u = (x + k, y + t) \} \]

\[ \frac{\partial y}{\partial x} = c(x,y) \]

\[ \frac{\partial x}{\partial y} = c(x,y) \]

\[ \frac{\partial y}{\partial x} = \frac{\partial x}{\partial y} \]

\[ A(x, y) = C(x, y) \]

\[ B(x, y) = C(x, y) \]

\[ S = \{ u = (x + k, y + t) \} \]

\[ c(x, y) = c(x, y) \]

\[ \frac{\partial x}{\partial y} = \frac{\partial y}{\partial x} \]

\[ A(x, y) = B(x, y) \]

\[ S = \{ u = (x + k, y + t) \} \]

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\[ A(x, y) = C(x, y) \]

\[ B(x, y) = C(x, y) \]

\[ S = \{ u = (x + k, y + t) \} \]

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\[ \frac{\partial x}{\partial y} = \frac{\partial y}{\partial x} \]

\[ A(x, y) = B(x, y) \]

\[ S = \{ u = (x + k, y + t) \} \]

\[ c(x, y) = c(x, y) \]

\[ \frac{\partial x}{\partial y} = \frac{\partial y}{\partial x} \]

\[ A(x, y) = C(x, y) \]

\[ B(x, y) = C(x, y) \]

\[ S = \{ u = (x + k, y + t) \} \]

\[ c(x, y) = c(x, y) \]

\[ \frac{\partial x}{\partial y} = \frac{\partial y}{\partial x} \]
Several independent variables

\[ PDE \quad F(x_1, \ldots, x_n, u_1, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}) = 0, \quad u_i = \frac{\partial u}{\partial x_i}, \text{ etc.} \]

Special case

\[ u = \phi(x_1, \ldots, x_n). \]

Initial data

\[
\begin{align*}
    x_1 &= x_{01} (x_1, \ldots, x_n) \\
    x_2 &= x_{02} (x_1, \ldots, x_n) \\
    \vdots
    x_n &= x_{0n} (x_1, \ldots, x_n) \\
    u &= u_0 (x_1, \ldots, x_n)
\end{align*}
\]

Monge form

\[
\begin{align*}
    x_1' &= x_1 + \tau F_{b_1}(x_1, \ldots, x_n, \tau, b_1, \ldots, b_n) \\
    x_2' &= x_2 + \tau F_{b_2}(x_1, \ldots, x_n, \tau, b_1, \ldots, b_n) \\
    \vdots
    x_n' &= x_n + \tau F_{b_n}(x_1, \ldots, x_n, \tau, b_1, \ldots, b_n) \\
    u' &= u + \tau (b_1 F_{b_1} + b_2 F_{b_2} + \cdots + b_n F_{b_n})
\end{align*}
\]

\[ F(x_1, \ldots, x_n, u, b_1, \ldots, b_n) = 0 \]

Characteristic equations

\[
\begin{align*}
    \frac{dx_1}{d\tau} &= F_{b_1} \\
    \vdots
    \frac{dx_n}{d\tau} &= F_{b_n} \\
    \frac{du}{d\tau} &= b_1 F_{b_1} + \cdots + b_n F_{b_n} \\
    \frac{db_1}{d\tau} &= -F_{x_1} - b_1 F_{u} \\
    \vdots
    \frac{db_n}{d\tau} &= -F_{x_n} - b_n F_{u}
\end{align*}
\]

Abbreviations of\n
\[
\begin{align*}
    \frac{dx}{d\tau} &= F_b \\
    \frac{dy}{d\tau} &= b F_b \\
    \frac{dk}{d\tau} &= -F_x - k F_u
\end{align*}
\]

Initial conditions

\[
\begin{align*}
    x_1 &= x_{01} (x_1, \ldots, x_n) \\
    \vdots
    x_n &= x_{0n} (x_1, \ldots, x_n) \\
    u &= u_0 (x_1, \ldots, x_n) \\
    b_1 &= b_{01} (x_1, \ldots, x_n) \\
    \vdots
    b_n &= b_{0n} (x_1, \ldots, x_n)
\end{align*}
\]
Lift of initial data

\[
\begin{align*}
\frac{\partial u_0}{\partial x_1} &= b_1 \frac{\partial x_{o1}}{\partial x_1} + \ldots + b_n \frac{\partial x_{on}}{\partial x_1} \\
\vdots \\
\frac{\partial u_0}{\partial x_{n-1}} &= b_1 \frac{\partial x_{o1}}{\partial x_{n-1}} + \ldots + b_n \frac{\partial x_{on}}{\partial x_{n-1}} \\
F(x_{o1}, \ldots, x_{on}, u_0, b_1, \ldots, b_n) &= 0
\end{align*}
\]

Known functions Unknwn functions of \( x_1, \ldots, x_{n-1} \)

Jacobian condition

\[
\Delta(x_1, \ldots, x_{n-1}, b_1, \ldots, b_n) =
\]

Solution of CE SPDE

\[
\begin{align*}
x_1 &= X_1(x_1, \ldots, x_{n-1}, x) \\
\vdots \\
x_n &= X_n \\
u &= u \\
b_1 &= P_1 \\
\vdots \\
b_n &= P_n
\end{align*}
\]

Similarly for \( x_1, \ldots, x_{n-1}, x \)

\& substitute \( u = U \).

\[
X_1(x_1, \ldots, x_{n-1}, 0) = x_{o1}(x_1, \ldots, x_{n-1}), \ldots
\]