Discrepancy Inequalities for Directed Graphs

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Joint work with Fan Chung

Discrepancy for Undirected Graphs

- G undirected graph
- ρ is the edge-density of G
- $\mu_2(A)$ is the second largest singular value of the adjacency matrix of G.

For a regular graph, $|E(S,T) - \rho|S||T|| \leq \mu_2(A)\sqrt{|S||T|}$

- d_u is the degree of u, $\mathrm{Vol}(S) = \sum_{u \in S} d_u$
- $\bar{\lambda} = \max\{\lambda_1, \lambda_{n-1}\}$ is the spectral gap of the Normalized Laplcacian

Theorem 1 (Chung 1997)

$$\left| E(S,T) - \frac{\mathrm{Vol}(S)\mathrm{Vol}(T)}{\mathrm{Vol}(G)} \right| \leq \bar{\lambda}\sqrt{\mathrm{Vol}(S)(T)}$$

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Discrepancy for Directed Graphs

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- $Vol_{in}(S) = |\{\{u,v\} \in E; v \in S\}|$
  $Vol_{out}(T) = |\{\{u,v\} \in E; u \in T\}|$
- D_{out} and D_{in} are the diagonal in- and out-degree matrices

Theorem 2 (Butler 2006)

$$\left| E(S,T) - \frac{\mathrm{Vol}_{out}(S)\mathrm{Vol}_{in}(T)}{|E(G)|} \right|$$

$$\leq \mu_2(D_{out}^{-1/2} A D_{in}^{-1/2}) \quad \sqrt{\mathrm{Vol}_{out}(S)\mathrm{Vol}_{in}(T)}$$

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Let *discrepancy of an undirected graph G* be the smallest number α such that $\forall S, T \subset V$:

$$| \, E(S,T) - \rho|S||T| \, | \leq \alpha\sqrt{|S||T|}$$

For α small, G must be almost regular

Theorem 3 (Bollobás and Nikiforov 2004)

There is a constant C such that for any undirected graph G on n vertices,

$$\mu_2(A) \leq C\alpha \log n$$

Further, there exists a family of regular graphs such that the logarithmic factor is necessary.

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There are constants C_1, C_2 such that for any directed graph, $\mu_2(D_{out}^{-1/2} A D_{in}^{-1/2}) \leq -C_1 \beta \log \beta + C_2 \beta$

We extend these ideas in the previous theorems by considering a random walk on the directed graph and the flow induced by its stationary distribution.

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Random Walks on Finite Strongly-Connected Aperiodic Directed Graphs

- G has edge-weights $w_{u,v} > 0 \iff \{u, v\} \in E$
- G is called *aperiodic* if the greatest common divisor of the length all closed walks is 1.
- G has an associated *probability transition matrix*, P given by

$$P_{u,v} = \frac{w_{u,v}}{\sum_x w_{u,x}}$$

- If G is strongly connected and aperiodic, then P is *ergodic* and has a *unique stationary distribution vector* ϕ which satisfies $\phi P = \phi$, $\phi(u) > 0$, $\sum_u \phi(u) = 1$

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For example, we do not consider... bipartite graphs, directed cycles, graphs with sources, nor graphs with sinks.

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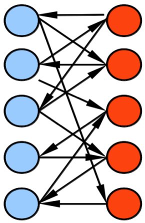

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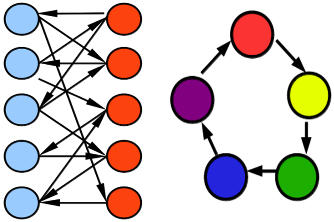

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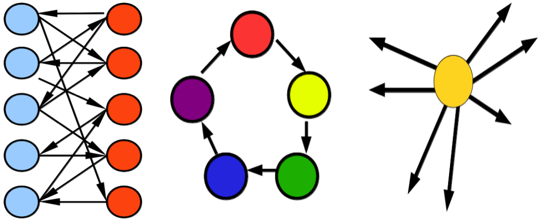

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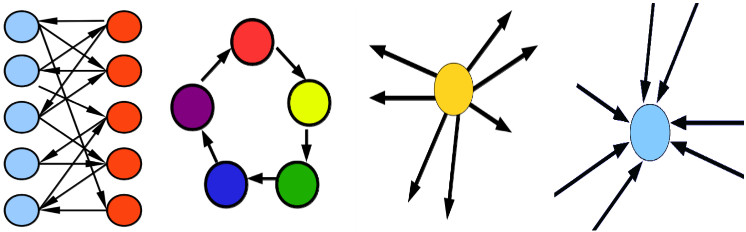
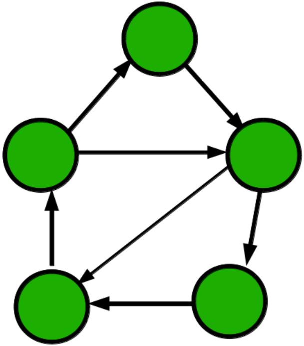
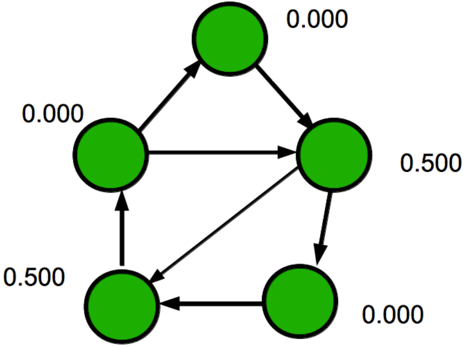


Example



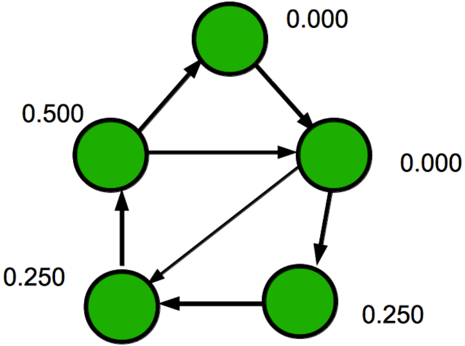
Example

Initial Configuration



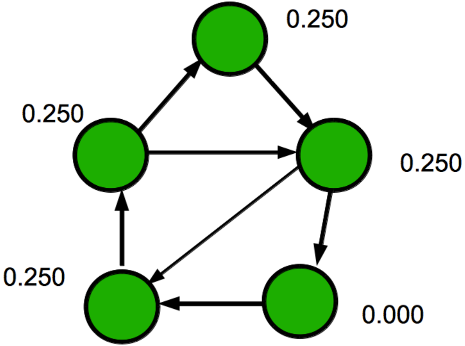
Example

After 1 Step



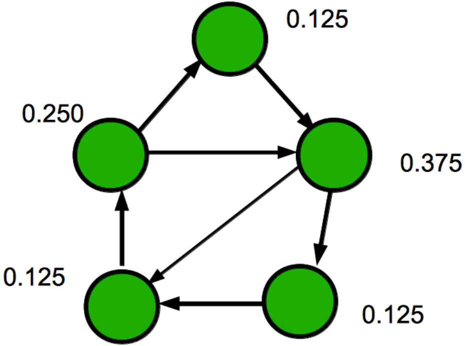
Example

After 2 Steps



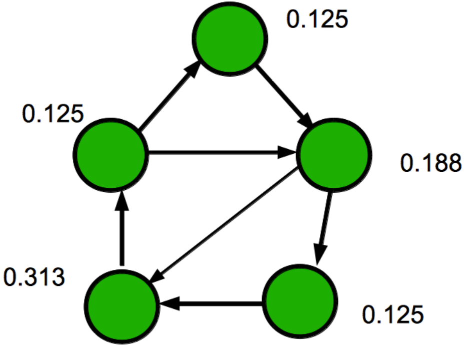
Example

After 3 Steps



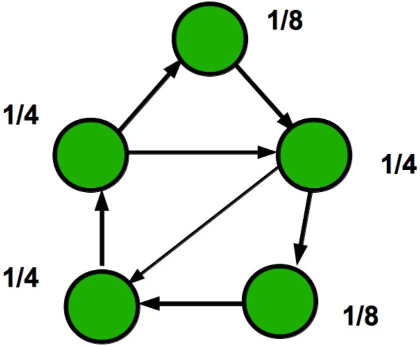
Example

After 4 Steps



Example

Stationary Distribution



ξ-circulation

ϕ induces a flow on G:

We define the *circulation* from vertex u to vertex v.

$$\xi(u,v) := \phi(u)P(u,v)$$

We extend ϕ and ξ to subsets $S, T \subset V(G)$:

$$\phi(S) := \sum_{s \in S} \phi(s) \qquad \xi(S,T) := \sum_{u \in S, v \in T} \xi(u,v).$$

Notice that $\xi(S,T)$ is not the same as $\xi(T,S)$.

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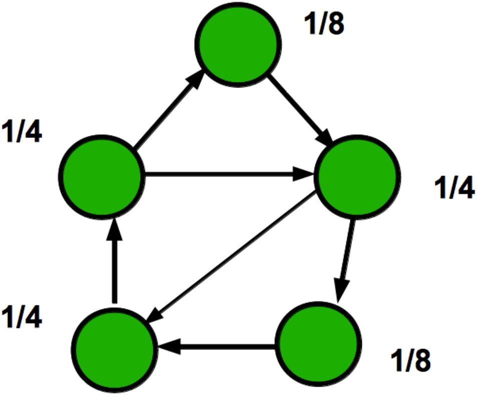
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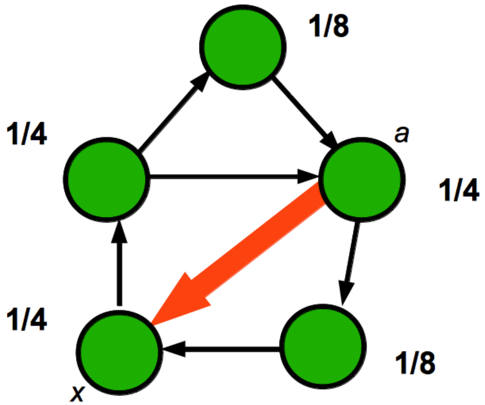
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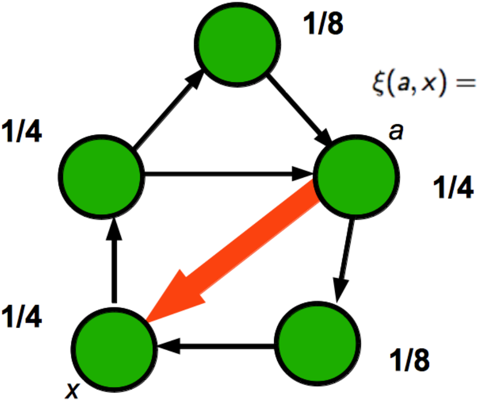

$$\xi(a,x) = \phi(a)P(a,x) = \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{8}$$

Two Types of Discrepancy

- *Discrepancy of subsets S and T of G:*
  $\mathrm{disc}(S,T) := |\,\xi(S,T) - \phi(S)\phi(T)\,|$
- *Discrepancy of directed graph G:*
  $\mathrm{disc}(G) :=$
  $\inf\{\alpha : \forall S,T \subset V(G), \mathrm{disc}(S,T) \leq \alpha\sqrt{\phi(S)\phi(T)}\}$
- *Skew-discrepancy of subsets S and T of G:*
  $\mathrm{disc}'(S,T) := |\,\xi(S,T) - \xi(T,S)\,|$
- *Skew-discrepancy of directed graph G:*
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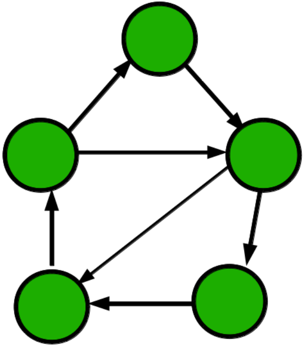

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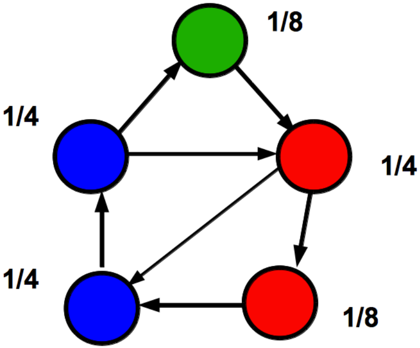

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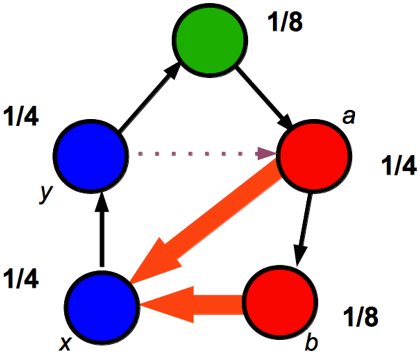

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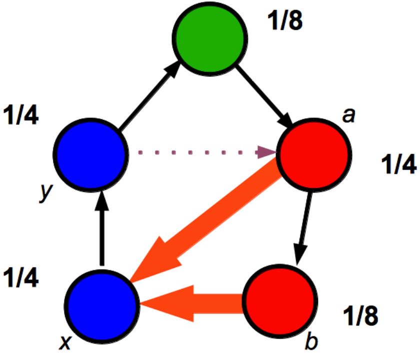

$$\begin{aligned} & \mathrm{disc}(R,B) \\ = & |\xi(R,B) - \phi(R)\phi(B)| \\ = & |\xi(a,x) + \xi(b,x) - \phi(R)\phi(B)| \\ = & |\frac{1}{8} + \frac{1}{8} - \frac{3}{8}\cdot\frac{1}{2}| = \frac{1}{16} \end{aligned}$$

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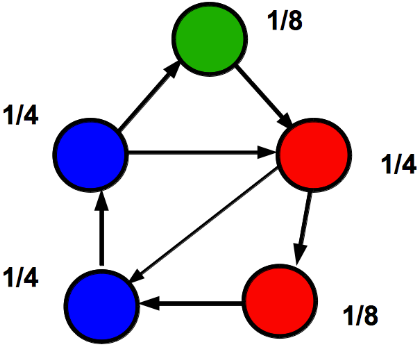

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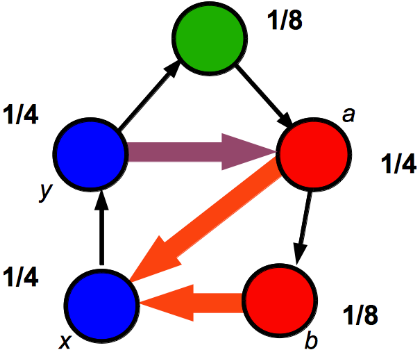

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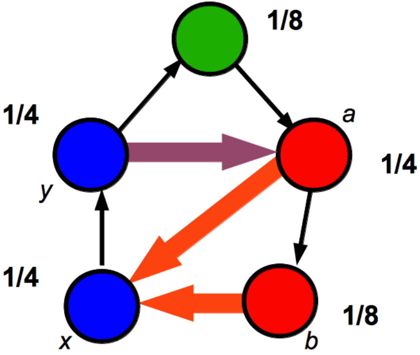

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Three Graph Theoretic Matrices

Normalized probability transition matrix, $\mathcal{P} := \Phi^{1/2} P \Phi^{-1/2}$ where

$$\Phi^{1/2} = \begin{pmatrix} \sqrt{\phi(u_1)} & 0 & 0 & \dots & 0 \\ 0 & \sqrt{\phi(u_2)} & 0 & \dots & 0 \\ 0 & 0 & \sqrt{\phi(u_3)} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sqrt{\phi(u_n)} \end{pmatrix}$$

Normalized Laplacian, $\mathcal{L} = I - \frac{\mathcal{P}+\mathcal{P}^T}{2} = I - \frac{\Phi^{1/2} P \Phi^{-1/2} + \Phi^{-1/2} P^T \Phi^{1/2}}{2}$
(Here, M^T denotes the transpose of a matrix.) [Chung 2005]

Skew-Symmetric Part of $\mathcal{P}$, $\mathcal{Z} = \frac{\mathcal{P}-\mathcal{P}^T}{2} = \frac{\Phi^{1/2} P \Phi^{-1/2} - \Phi^{-1/2} P^T \Phi^{1/2}}{2}$

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Main Results: Overview

- $\mathrm{disc}(S,T)$ and $\mathrm{disc}(G)$ are strongly related to σ_2, the second largest singular value of $\mathcal{P}$
- $\mathrm{disc}(S,\bar{S})$ is strongly related to the spectral gap of $\mathcal{L}$, the normalized Laplacian of G
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Upper bounds for $\mathrm{disc}(S, T)$

Theorem 6

Let G be a directed graph with a unique stationary distribution, ϕ. Let $\phi(S)$ denote the the total stationary distribution on S, and $\xi(S, T)$ denote the flow from S to T under ϕ, Then, for any subset of vertices $S, T \subset V(G)$, we have

$$\begin{aligned} \mathrm{disc}(S, T) &:= \left| \xi(S, T) - \phi(S)\phi(T) \right| \\ &\leq \sigma_2(\mathcal{P})\sqrt{\phi(S)\phi(\bar{S})\phi(T)\phi(\bar{T})} \end{aligned}$$

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Let $\mathrm{disc}(G) = \alpha$ be the least α' such that for every subset of vertices $S, T \subset V(G)$,

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Then,

$$\alpha \leq \sigma_2(\mathcal{P}) \leq 96\alpha - 969\alpha \log \alpha$$

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For the upper bound, given two subsets, A, B, consider the bilinear form

$$(\Phi^{1/2}\mathbf{1}_A - (\mathbf{1}_A^T\Phi\mathbf{1})\Phi^{1/2}\mathbf{1}))^T\mathcal{P}(\Phi^{1/2}\mathbf{1}_B^T - (\mathbf{1}_B^T\Phi\mathbf{1})\Phi^{1/2}\mathbf{1})$$

For the lower bound, we expand the techniques of [Bollabás and Nikiforov 2004], [Bilu and Linial 2004], and [Butler 2006] by using discretized vectors approximating the left and right singular vectors of $\mathcal{P}$.

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$$\mid \xi(S,\bar{S}) - \phi(S)(\phi(\bar{S})) \mid \leq \bar{\lambda}(\phi(S)\phi(\bar{S}))$$

where $\bar{\lambda} = \max_{i\neq 0} |1-\lambda_i|$ *and* $\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{n-1}$ *are eigenvalues of* $\mathcal{L}$.

This is closely related to the *Cheeger constant* of a directed graph [Chung 2005].

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Using the previous notions of the directed graph G, stationary distribution ϕ, and the flow ξ.

Let ψ_{max} be the largest singular value of $\mathcal{Z}$.

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Observe that $\psi_{max} \geq \frac{Im(f^H \mathcal{P} f)}{f^H f}$

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Define $\mathcal{L}' = \mathcal{L} + \mathrm{i}\mathcal{Z}$ to be the *Quantum Laplacian*

$\mathcal{L}'$ is Hermitian

(Let $\hat{q}$ denote the smallest non-zero eigenvalue, and q_{max} be the largest eigenvalue of $\mathcal{L}'$.)

Theorem 12

$$\begin{aligned} q_{max} &\leq 96\mathrm{disc}(G) - 969\mathrm{disc}(G)\log\mathrm{disc}(G) \\ &+ 48\sqrt{2}\mathrm{disc}'(G) - 485\sqrt{2}\mathrm{disc}'(G)\log\mathrm{disc}'(G)) \end{aligned}$$

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Thank You!

- R. Ahlswede, A. Winter. Strong converse for identification via quantum channels. *IEEE Transactions on Information Theory* **48** (2002), no. 3, 569–579.
- Y. Bilu, N. Linial. Constructing Expander Graphs by 2-Lifts and Discrepancy vs. Spectral Gap. *Annual IEEE Symposium on Foundations of Computer Science, 45th Annual IEEE Symposium on Foundations of Computer Science* (FOCS'04), (2004) 404–412.
- B. Bollobás, V. Nikiforov. Hermitian matrices and graphs: singular values and discrepancy, Discrete Mathematics, **285** (2004) 17–32.
- S. Butler, Using discrepancy to control singular values for nonnegative matrices. *Linear Algebra and its Applications* **419** (2006) 486–493.
- F. Chung, *Spectral Graph Theory*, AMS Publications, Providence, RI (1997).
- F. Chung, Laplacians and the Cheeger inequality for directed graphs, *Annals of Combinatorics*, 9 (2005), 1–19.