MATH 190A
WINTER 2021

Problem Set 2

Due by 2359 (11:59 PM) on Sunday January 17. Hand in via Gradescope.

You may discuss these problems among yourselves, but your final submitted solutions must be written by you alone, in your own words, without any other solution in front of you.

We give the following definitions in case they have not yet been covered in class:—

- For two topological spaces $(X, \mathcal{U}_X)$ and $(Y, \mathcal{U}_Y)$, a function $f\colon X \to Y$ is continuous if $f^{-1}(U) \in \mathcal{U}_X$ for all $U \in \mathcal{U}_Y$.
- In $(X, \mathcal{U}_X)$, a sequence $x_1, x_2, \dots,$ in X converges to $y \in X$ if for all $U \in \mathcal{U}_X$, $y \in U$, there exists $N \geq 1$ such that $x_n \in U$ for all $n \geq N$.
- The closure of $A \subseteq X$ consist of all points $x \in X$ such that for all $U \in \mathcal{U}_X$, $x \in U$, we have $U \cap A \neq \emptyset$.

1. Which of the following sequences converge in the 5-adic metric on $\mathbb{Q}$?

(a) 2021, 2022, 2023, 2024, $\dots$;

(b) 2021, 20021, 200021, 2000021, $\dots$?

2. Consider the space

$$X = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\} \setminus \{(1, 0)\}$$

(using the Euclidean metric $d(x, y) = \|x - y\|$ on $\mathbb{R}$, say). Is it homeomorphic to $(0, 1)$? Justify your answer.

3. (a) Describe all the topologies on $\{1, 2\}$.

(b) For each topology, describe concisely all the infinite sequences $x_1, x_2, \dots$ and all the elements $y \in \{1, 2\}$ such that $x_n \to y$ converges.

(c) For each topology and each $x \in \{1, 2\}$, determine the closure $\overline{\{x\}}$.

(d) For each pair of topological spaces X and Y, describe concisely which of the 4 functions $f\colon \{1, 2\} \to \{1, 2\}$ are continuous when considered as maps from X to Y.

4. Consider the set $X = \mathbb{N} \cup \{\infty\}$, where $\mathbb{N} = \{1, 2, \dots\}$ and ∞ is another arbitrary symbol. Define $\mathcal{U}$ to consist of the following subsets of X:

- $U = S \cup \{\infty\}$ where $S \subseteq \mathbb{N}$ is co-finite (i.e., $S = \mathbb{N} \setminus T$ for some finite set $T \subseteq \mathbb{N}$); or
- any set $U \subseteq \mathbb{N}$.

(a) Briefly, verify that this is a topology on X.

(b) Let Y be a topological space. Prove that $f\colon X \to Y$ is a continuous function if and only if

$$f(1),\ f(2),\ f(3),\ \cdots \to f(\infty)$$

converges in Y.

5. Let X be a metric space[1]. If $A \subseteq X$, the *interior* of A is defined as

$$\operatorname{int}(A) = \{x \in A \colon \exists U \subseteq A \text{ open such that} x \in U\}.$$

(a) Show that $\operatorname{int}(A) = \left(\overline{A^c}\right)^c$, where $B^c = X \setminus B$ denotes the complement of a set B.

(b) Given two sets $A, B \subseteq X$, show that $A \subseteq \overline{B}$ if and only if $\overline{A} \subseteq \overline{B}$.

(c) Write $\operatorname{cl}(A) = \overline{A}$. Prove that for any $A \subseteq X$,

$$\operatorname{cl}(\operatorname{int}(\operatorname{cl}(\operatorname{int}(A)))) = \operatorname{cl}(\operatorname{int}(A))$$

and

$$\operatorname{int}(\operatorname{cl}(\operatorname{int}(\operatorname{cl}(A)))) = \operatorname{int}(\operatorname{cl}(A)).$$

[Hence, there are at most 7 distinct sets, including A itself, that can be obtained from A by repeatedly taking closures and interiors.]

[1] or topological space; it doesn't matter for this question.