MATH 190A
WINTER 2021

Problem Set 3

Due by 2359 (11:59 PM) on Sunday January 24. Hand in via Gradescope.

You may discuss these problems among yourselves, but your final submitted solutions must be written by you alone, in your own words, without any other solution in front of you.

1. Consider $X = \{0,1\}^{\mathbb{N}}$, the set of all $\{0,1\}$-valued sequences $(a_1, a_2, \dots)$.

Define sets $B \subseteq X$ by fixing the first k coordiantes. That is, for $k \geq 0$ and elements $b_1, \dots, b_k \in \{0,1\}$, set

$$B\big((b_i)_{i=1}^k\big) = \big\{(a_1, a_2, \dots) \in X \colon a_1 = b_1, \dots, a_k = b_k\big\}.$$

Finally let

$$\mathcal{B} = \big\{B\left((b_i)_{i=1}^k\right) : k \geq 0,\ b_1, \dots, b_k \in \{0,1\}\big\}$$

be the collection of all such sets B.

(a) Show that $\mathcal{B}$ is a basis for some topology $\mathcal{U}$ on X.

(b) Show that the map

$$X \to [0,1]$$
$$(a_1, a_2, \dots) \mapsto \sum_{i=1}^{\infty} 2^{-i} a_i$$

is continuous with respect to this topology.

(c) Is this map a homeomorphism? Justify your answer.

2. Which of the following pairs of subspaces of $\mathbb{R}$, considered with the subspace topology, are homeomorphic? Justify your answers carefully.

(a) $X = (0,1]$ and $Y = (0,1) \cup \{2\}$;

(b) $X = \{1/n \colon n \geq 1\} \cup \{0\}$ and $Y = \{\pm 1/n \colon n \geq 1\} \cup \{0\}$;

(c) $X = \{1/n \colon n \geq 1\} \cup \{0\}$ and

$$Y = \{1/n \colon n \geq 1\} \cup \{0\} \cup \{2 + 1/n \colon n \geq 1\} \cup \{2\}?$$

3. Consider

$$\mathcal{B} = \big\{[a,b) \colon a, b \in \mathbb{Q},\ a \leq b\big\}$$

You should verify for yourself that this generates a topology $\mathcal{U}_1$ on $\mathbb{R}$, but you do not need to prove this. Write $\mathcal{U}$ for the usual (metric) topology on $\mathbb{R}$.

(a) Prove that the identity map $(\mathbb{R}, \mathcal{U}_1) \to (\mathbb{R}, \mathcal{U})$ is continuous.

(b) Characterize sequences $(x_n)_{n\geq 1}$ in $\mathbb{R}$ such that (i) $x_n \to 0$, or (ii) $x_n \to \sqrt{2}$.