MATH 190A
WINTER 2021

Problem Set 5

Due by 2359 (11:59 PM) on Sunday February 7. Hand in via Gradescope.

You may discuss these problems among yourselves, but your final submitted solutions must be written by you alone, in your own words, without any other solution in front of you.

1. Prove that the quotient topology defined in lectures is indeed a topology.

2. Let $\mathbb{R}$ denote the reals with the usual metric topology. Define an equivalence relation $\sim$ on $\mathbb{R}$ by

$$x \sim y \Leftrightarrow x - y \in \mathbb{Q}.$$

The quotient space $\mathbb{R}/\sim$ is often denoted $\mathbb{R}/\mathbb{Q}$.

(a) Verify that $\sim$ is indeed an equivalence relation, and describe your favorite two *distinct* equivalence classes in $\mathbb{R}/\sim$.

(b) Prove that the quotient topology on $\mathbb{R}/\sim$ is indiscrete; that is, the only open sets are $\emptyset$ and $\mathbb{R}/\sim$.

[In particular, a quotient of a metric space need not be a metric space.]

3. Let X be a topological space, $\sim$ and equivalence relation on X and $X/\sim$ the quotient space with the quotient topology. Let $\pi\colon X \to X/\sim$ denote the quotient map. For each of the following statements give either a proof or a counterexample.

(a) If $U \subseteq X$ is open in X then $\pi(U)$ is open in $X/\sim$.

(b) If $U \subseteq X$ is open in X then $\left\{y \in X/\sim \colon \pi^{-1}(y) \subseteq U\right\}$ is open in $X/\sim$.

4. Let

$$X = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 1\} \subseteq \mathbb{R}^2$$

denote the unit (filled) unit disc, with the subspace topology. Let $\mathbb{R}^2$ have the Euclidean metric.

Define an equivalence relation $\sim$ on X as follows: let

$$S = \left\{(x, y) \in X \colon x^2 + y^2 = 1\right\} \subseteq X$$

i.e., S is the boundary of the unit disc. Then for $a, b \in X$, set

$$a \sim b \Leftrightarrow a = b \text{ or } a, b \in S.$$

You do not need to prove that this is an equivalence relation. Write $\pi\colon X \to X/\sim$ for the quotient map and $[x]$ for the equivalence class of $x \in X$.

(a) Describe the equivalence classes of $\sim$.

(b) By considering the following map

$$f\colon X \to \mathbb{R}^3$$
$$(a, b) \mapsto \left(2a\sqrt{1 - a^2 - b^2},\ 2b\sqrt{1 - a^2 - b^2}, 2(a^2 + b^2) - 1\right),$$

prove that there exists a corresponding *continuous bijection*

$$F\colon X/{\sim} \to \{(x,y,z) \in \mathbb{R}^3\colon x^2+y^2+z^2=1\} \subseteq \mathbb{R}^3$$

from $X/{\sim}$ to the unit sphere in $\mathbb{R}^3$.
[You may use without proof that f is continuous.]

(c) Prove that the following sets are a basis for the quotient topology on $X/{\sim}$:

$$\mathcal{B} = \Big\{\{[(x',y')]\colon (x',y') \in B_\varepsilon((x,y))\}\colon (x,y) \in X,\ x^2+y^2<1,\ 0<\varepsilon<1-\sqrt{x^2+y^2}\Big\} \cup \Big\{\{[(x,y)]\colon (x,y)\in X,\ \sqrt{x^2+y^2}>1-\varepsilon\}\colon 0<\varepsilon<1\Big\}.$$

[**Hint:** you are trying to show that (i) the sets $B \in \mathcal{B}$ are open in $X/{\sim}$, and (ii) for any open set U in $X/{\sim}$ and any $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$. For (ii), consider separately the cases that $x \in \pi(S)$ and $x \notin \pi(S)$.]

(d) Prove that F is a homeomorphism.
[**Hint:** show that for all $B \in \mathcal{B}$, $F(B)$ is an open subset of the unit sphere, and explain why this implies that F^{-1} is continuous.]