MATH 190A
WINTER 2021

PROBLEM SET 9

Due by 2359 (11:59 PM) on Sunday March 7. Hand in via Gradescope.

You may discuss these problems among yourselves, but your final submitted solutions must be written by you alone, in your own words, without any other solution in front of you.

1. Let X be a compact toplogical space. Suppose $C_1, C_2, C_3, \dots$ are closed subsets of X such that $C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$. Suppose moreover that C_i are all non-empty.

Prove that $\bigcap_{i=1}^{\infty} C_i$ is non-empty.

2. Let X, Y be two topological spaces and suppose they are both sequentially compact. Prove that $X \times Y$ is sequentially compact.

[Note: you may not assume X,Y are metric spaces, so an argument that tries to show X,Y are compact will not succeed.]

3. Let X be a compact metric space. Which of the following statements are true? Justify your answers.

(a) For any $\varepsilon > 0$ there exists a finite cover of X by closed balls of radius ε; that is, for some $x_1, \dots, x_n \in X$ we have

$$X = B_\varepsilon[x_1] \cup B_\varepsilon[x_2] \cup \dots \cup B_\varepsilon[x_n].$$

(Note these are closed balls, $B_\varepsilon[y] = \{x \in X : d(x, y) \le \varepsilon\}$.)

(b) For any cover of X by closed balls of radius $\varepsilon > 0$, there exists a finite subcover: that is, if

$$X = \bigcup_{i \in I} B_\varepsilon[x_i]$$

for some collection $(x_i)_{i \in I}$, then there exist $i_1, \dots, i_m \in I$ such that

$$X = B_\varepsilon[x_{i_1}] \cup B_\varepsilon[x_{i_2}] \cup \dots \cup B_\varepsilon[x_{i_m}].$$

4. Let X be a compact metric space and let $(U_i)_{i \in I}$ be an open cover of X.

Prove that there exists $\varepsilon > 0$ (a *Lebesgue number* for the covering) such that for all $x \in X$ there exists $i \in I$ such that $B_\varepsilon(x) \subseteq U_i$.

[**Hint**: you may wish to consider the open sets of the form

$$U_{i,n} = \{x \in X : B_{1/n}(x) \subseteq U_i\}$$

for $i \in I$ and $n \in \mathbb{N}$.]

5. Consider the space $X = \{0, 1\}^{\mathbb{N}}$ (infinite sequences of 0s and 1s) with the following metric:

$$d\big((x_i)_{i \in \mathbb{N}}, (y_i)_{y \in \mathbb{N}}\big) = \sum_{i=1}^{\infty} 2^{-i} |x_i - y_i|.$$

You may use without proof that this is a metric (although you should convince yourself why this is so).

(a) Prove that the metric topology induced by this metric, coincides with the topology from Pset 3 Q1.

(b) Prove directly that X with this metric is sequentially compact.

(c) Suppose $f\colon \{0,1\}^{\mathbb{N}} \to \{0,1\}$ is a function. Suppose that for all $x \in \{0,1\}^{\mathbb{N}}$ there exists $m \in \mathbb{N}$ such that

$$\forall y \in \{0,1\}^{\mathbb{N}} \text{ such that } y_1 = x_1, \ldots, y_m = x_m \colon \text{ we have } f(x) = f(y).$$

Prove that there exists a global value $M \in \mathbb{N}$ (independent of x) such that for all $x \in \{0,1\}^{\mathbb{N}}$,

$$\forall y \in \{0,1\}^{\mathbb{N}} \text{ such that } y_1 = x_1, \ldots, y_m = x_M \colon \text{ we have } f(x) = f(y).$$

[This has a not-entirely-obvious interpretation as the statement: if a computer program terminates on every possible infinite input string, then it terminates after some *globally bounded* amount of time.]