Due by the beginning of class on Friday 11th October (hand in via Gradescope).

1. For each of the following, determine whether it is an inner product on \( \mathbb{R}^2 \).
   (a) \( \langle (x, y), (x', y') \rangle = xx' - yy' \).
   (b) \( \langle (x, y), (x', y') \rangle = xx' + xy' + yx' + 2yy' \).
   (c) \( \langle (x, y), (x', y') \rangle = xx' + xy' + yx' + yy' \).

2. Let \( V \) be a finite dimensional inner product space, with inner product \( \langle - , - \rangle \) as usual.
   (a) Suppose \( \phi : V \rightarrow V \) a linear map. Define a new operation \( \langle - , - \rangle_1 : V \times V \rightarrow F \) by
      \[ \langle v, w \rangle_1 = \langle \phi(v), \phi(w) \rangle. \]
      Show that if \( \phi \) is invertible then this is an inner product on \( V \).
   (b) In the set-up of (a), show that if \( \phi \) is not invertible then \( \langle - , - \rangle_1 \) is not an inner product.
   (c) Conversely, suppose \( \langle - , - \rangle_2 \) is yet another inner product on \( V \). Show that there is an invertible linear map \( \psi : V \rightarrow V \) such that
      \[ \langle v, w \rangle_2 = \langle \psi(v), \psi(w) \rangle. \]

3. Consider the vectors
   \[ v_1 = (1.1, 1.1, 1.1, 1.1) \]
   \[ v_2 = (3.3, 3.3, 1.1, 1.1) \]
   \[ v_3 = (6.6, 4.4, 2.2, 0) \]
   \[ v_4 = (10, 5.4, 3.2, -0.9) \]
   in \( \mathbb{R}^4 \), which carries the usual dot product.
   You may assume that
   \[
   \begin{pmatrix}
   1.1 & 3.3 & 6.6 & 10 \\
   1.1 & 3.3 & 4.4 & 5.4 \\
   1.1 & 1.1 & 2.2 & 3.2 \\
   1.1 & 1.1 & 0 & -0.9 \\
   \end{pmatrix}
   =
   \begin{pmatrix}
   0.5 & 0.5 & 0.5 & 0.5 \\
   0.5 & 0.5 & -0.5 & -0.5 \\
   0.5 & -0.5 & 0.5 & -0.5 \\
   0.5 & -0.5 & -0.5 & 0.5 \\
   \end{pmatrix}
   \begin{pmatrix}
   2.2 & 4.4 & 6.6 & 8.8 \\
   0 & 2.2 & 4.4 & 6.6 \\
   0 & 0 & 2.2 & 4.4 \\
   0 & 0 & 0 & 0.2 \\
   \end{pmatrix}.
   \]
   Show that there exist coefficients \( a_1, a_2, a_3 \in \mathbb{R} \) such that
   \[ \|v_4 - a_1v_1 - a_2v_2 - a_3v_3\| \leq 0.2. \]
   [Please tell me if you spot a mistake in these numbers.]

4. Let \( \mathcal{P}_{\leq 4} \) denote the vector space of real-valued polynomials of degree \( \leq 4 \):
   \[ \mathcal{P}_{\leq 4} = \{ p(X) = a_0 + a_1X + \cdots + a_4X^4 : a_0, \ldots, a_4 \in \mathbb{R} \}. \]
You may use without proof that $B = 1, X, X^2, \ldots, X^4$ and $B' = 1, 1 + X, (1 + X)^2, \ldots, (1 + X)^4$ are two basis for $P_{\leq 4}$, and that

$$\phi: p(X) \mapsto \frac{dp}{dX}$$

and

$$\psi: p \mapsto (X \mapsto p(X + 1))$$

are linear maps $P_4 \to P_4$. (So e.g. $\psi(3 + 4X) = 3 + 4(X + 1) = 7 + 4X$.)

Write down:

(a) $M(\phi, B, B)$;
(b) $M(\psi, B, B)$;
(c) $M(\text{id}, B', B)$;
(d) $M(\text{id}, B, B')$;
(e) $M(\phi, B, B')$.

For a subspace $U \subseteq V$, write

$$U^\perp = \{ \phi \in V^*: \phi(u) = 0 \ \forall u \in U \} \subseteq V^*.$$

Similarly, if $W \subseteq V^*$, write

$$W^\perp = \{ u \in V: \phi(u) = 0 \ \forall \phi \in W \} \subseteq V.$$

5. Prove that if $V$ is finite-dimensional (and so we can identify $V$ with $V^{**}$) then $(U^\perp)^\perp = U$.

6. Let $V$ and $W$ be vector spaces, let $f: V \to W$ be a linear map, and let $f^*: W^* \to V^*$ be the dual map.
   (a) Prove that $\ker(f^*) = (\text{im } f)^\perp$.

   Now assume that $V, W$ are finite-dimensional (and so we can identify $V$ with $V^{**}$ and $W$ with $W^{**}$).

   (b) Prove that $\phi^{**} = \phi$.

   (c) Prove that $(\text{im } f^*)^\perp = \ker f$.

   (d) Prove that $(\ker f^*)^\perp = \text{im } f$.

   (e) Prove that $\text{im } f^* = (\ker f)^\perp$. 