1. Suppose $V$ is a finite-dimensional vector space, $\phi: V \rightarrow V$ is a nilpotent linear map, $m \geq 1$, $n_1, \ldots, n_m \geq 1$ are integers ($n_1 + \cdots + n_m = \dim V$) and $v_{i,j}$ ($1 \leq i \leq m$, $1 \leq j \leq n_i$) are some basis for $V$ such that

$$\phi(v_{i,j}) = \begin{cases} v_{i,j-1} & : j > 1 \\ 0 & : j = 1 \end{cases}$$

as guaranteed by Theorem 4.2.7.

(a) Prove that $\dim \ker(\phi^k) = \sum_{i=1}^{m} \min(k, n_i)$.

(b) Hence, for each integer $r \geq 1$, find an expression for $|\{i: 1 \leq i \leq m, n_i = r\}|$ (the number of blocks of size $r$) in terms of the quantities $\dim \ker(\phi^k)$.

[In other words, this shows nilpotent normal form is unique up to permuting the blocks.]

2. Consider the space $V$ of complex sequences $(a_0, a_1, a_2, \ldots)$ satisfying

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_d a_{n-d}$$

for all $n \geq d$, where $d \geq 1$ is an integer and $r_1, \ldots, r_d \in \mathbb{C}$ are fixed constants.

Show that there is a basis for $V$ consisting of sequences of the form $a_n = \lambda^n \binom{n}{\ell}$ for some $\lambda \in \mathbb{C}$ and $\ell \geq 0$.

[Hint: let $\phi: V \rightarrow V$ denote the (infinite) left shift, as in Pset 6 Q6. One approach is to consider $v = (a_0, a_1, \ldots) \in G(\lambda, \phi)$ and find an explicit formula for $a_n$.]

3.(a) Suppose $V$ is an inner product space with inner product $\langle -, - \rangle$ and associated norm $\| \cdot \|$. Prove that the parallelogram law holds for $\| \cdot \|$: that is, for all $v, w \in V$ we have

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2.$$

(b) For any $n \geq 1$ and $1 \leq p \leq \infty$, $p \neq 2$, find vectors $v, w \in \mathbb{R}^n$ such that

$$\|v + w\|^p_p + \|v - w\|^p_p \neq 2\|v\|^p_p + 2\|w\|^p_p.$$

4.(a) Suppose $V$ is a finite-dimensional real vector space and $\| \cdot \|$ is a norm on $V$ obeying the parallelogram law: that is,

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$$

for all $v, w \in V$. Prove that there is an inner product $\langle -, - \rangle$ on $V$ such that $\|v\| = \sqrt{\langle v, v \rangle}$.

[Hint: consider the quantity $\frac{1}{2} (\|v + w\|^2 - \|v - w\|^2)$.

(b) Prove the same statement as (i) assuming now that $V$ is a complex vector space.

[Hint: it may now help to build an expression from the quantities $\|v \pm w\|^2$ and $\|v \pm iw\|^2$.]
5. Let $V = \mathbb{R}^n$, and identify $V^* = \mathbb{R}^n$ in the usual way; i.e., a vector $(a_1, \ldots, a_n) \in \mathbb{R}^n$ corresponds to a linear map $\mathbb{R}^n \to \mathbb{R}$,

$$(x_1, \ldots, x_n) \mapsto a_1x_1 + \cdots + a_nx_n.$$ 

For each of the norms $\| \cdot \|_\infty$ and $\| \cdot \|_1$ on $\mathbb{R}^n$, determine carefully the value of the corresponding dual norms $\|(a_1, \ldots, a_n)\|_\infty^*$, $\|(a_1, \ldots, a_n)\|_1^*$ for all $(a_1, \ldots, a_n) \in \mathbb{R}^n \cong V^*$. 