Math 200a (Fall 2016) - Homework 7

Professor D. Rogalski

Posted November 14–Due **Wed. November 23** at 3pm

1 Reading

Read Chapter 8.

2 Exercises to submit on Wed. November 23

Exercise 1. Let R be a commutative ring, and let $I = (r_1, \dots, r_n)$ be a nonzero finitely generated ideal of R. Prove that there is an ideal J of R which is maximal among ideals which do not contain I.

Exercise 2. Let R be a commutative ring. Prove that the set of prime ideals of R has a minimal element with respect to inclusion (possibly the zero ideal). Such a prime ideal is called a *minimal prime* of R. (Hint: the trick to prove this result is to use Zorn's lemma with the following poset structure on the set of prime ideals: $P \leq Q$ means $Q \subseteq P$; thus a maximal element in this poset is a minimal prime.)

Exercise 3. Let R be a commutative ring. The *nilradical* is the ideal consisting of all nilpotent elements of R (you showed this is an ideal of R on the previous homework). The intersection of all prime ideals of the ring R is an ideal called the *prime radical*. In the steps of this problem you will prove that the nilradical and the prime radical are equal, so they are just two names for the same thing.

(a). Show that if $x \in R$ is nilpotent, then x is contained in every prime ideal of R.

(b). Prove that a proper ideal P of R is prime if and only if it satisfies the following condition: for every pair of ideals I and J of R with $P \subseteq I$ and $P \subseteq J$, if $IJ \subseteq P$ then either $I = P$ or $J = P$.

(c). Show that if $x \in R$ is not nilpotent, then there is some prime ideal P not containing x. (Hint: Let $X = \{1, x, x^2, \dots, \}$ which by assumption does not contain 0. Use Zorn to show there is some ideal P which is maximal among the set S of ideals I of R such that $I \cap X = \emptyset$. Use part (b) to show that P is prime). Conclude that the nilradical equals the prime radical.

Exercise 4. Let R be a commutative ring. The ring of formal Laurent series over R is the ring $R((x))$ given by

$$R((x)) = \{\sum_{n \geq N}^{\infty} a_n x^n | a_n \in R, N \in \mathbb{Z}\}.$$

Note that this is similar to the power series ring $R[[x]]$, except that Laurent series are allowed to include finitely many negative powers of x. The product and sum in this ring are defined similarly as for power series.

(a). Prove that if F is a field, then $F((x))$ is a field. (Hint: you may want to derive this from the result you proved on the previous homework that an element in the power series ring $F[[x]]$ is a unit in $F[[x]]$ if and only if it has nonzero constant term).

(b). Prove that if F is a field, then $F((x))$ is isomorphic to the field of fractions of $F[[x]]$. (Hint: use the universal property of the localization to show there is a map from the field of fractions to $F((x))$, then show it is surjective).

(c). Show that $\mathbb{Q}((x))$ is *not* the field of fractions of its subring $\mathbb{Z}[[x]]$. (Hint: consider the power series representation of e^x.)

Exercise 5. Recall that when D is a squarefree integer, then the *ring of integers* in the field $\mathbb{Q}(\sqrt{D}) = \{x + y\sqrt{D} | x, y \in \mathbb{Q}\}$ is the subring $\mathcal{O} = \{a + b\omega | a, b \in \mathbb{Z}\}$ of $\mathbb{Q}(\sqrt{D})$, where $\omega = \sqrt{D}$ if D is congruent to 2 or 3 modulo 4, while $\omega = (1 + \sqrt{D})/2$ if D is congruent to 1 modulo 4. The field $\mathbb{Q}(\sqrt{D})$ has the norm function $N(a + b\sqrt{D}) = |a^2 - Db^2|$, which is multiplicative, i.e. $N(z_1 z_2) = N(z_1)N(z_2)$ for $z_1, z_2 \in \mathbb{Q}(\sqrt{D})$.

(a). Consider the ring of integers $\mathcal{O}$ in $\mathbb{Q}(\sqrt{D})$. Suppose that for every $z \in \mathbb{Q}(\sqrt{D})$, there exists an element $y \in \mathcal{O}$ such that $N(z - y) < 1$. Prove that $\mathcal{O}$ is a Euclidean domain. (Hint: follow the method of proof we used to show that $\mathbb{Z}[i]$ is a Euclidean domain).

(b). Show that the ring of integers $\mathcal{O}$ is a Euclidean domain when $D = -2, 2, -3, -7$, or -11. (In each case show that part (a) applies).

Exercise 6. A *Bezout domain* is an integral domain R in which every ideal generated by 2 elements is principal; that is, given $a, b \in R$ we have $(a, b) = (d)$ for some d.

(a). Prove that an integral domain R is a Bezout domain if and only if every pair of elements a, b has a GCD $d \in R$ such that $d = ax + by$ for some $x, y \in R$.

(b). Prove that every finitely generated ideal of a Bezout domain is principal.

(c). Prove that R is a PID if and only if R is both a UFD and a Bezout domain. (Hint: If R is a UFD and Bezout, given a nonzero ideal I, choose $0 \neq a \in I$ with a minimal number of irreducibles in its factorization. Given an arbitrary $b \in I$ consider the ideal (a, b).)

Exercise 7. Recall that a commutative ring R is *local* if it has a unique maximal ideal M. We proved in class that R is local with maximal ideal M if and only if every element of $R - M$ is a unit in R.

(a). Let P be a prime ideal of R. Let $X = R - P$ be the set of elements in R which are not in P. Consider the localization RX^{-1}. Show that RX^{-1} is a local ring, with unique maximal ideal $PX^{-1} = \left\{ \frac{r}{x} \,\middle|\, r \in P, x \in X \right\}$.

(b). Note that R/P is a domain, since P is prime. Show that RX^{-1}/PX^{-1} is isomorphic to the field of fractions of R/P.