Math 200b (Winter 2016) - Homework 1

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The exercises can be found in Dummit and Foote: p. 519, ##1, 3, 5 and pp. 529-531, ## 1, 3, 5, 10, 14, 16, 20.

Exercise 1. Show that $p(x) = x^3 + 9x + 6$ is irreducible in $\mathbb{Q}[x]$. Let θ be a root of $p(x)$. Find the inverse of $1 + \theta$ in $\mathbb{Q}(\theta)$.

Exercise 2. Show that $x^3 + x + 1$ is irreducible over $\mathbb{F}_2$ and let θ be a root. Compute the powers of θ in $\mathbb{F}_2(\theta)$.

Exercise 3. Suppose α is a rational root of a monic polynomial in $\mathbb{Z}[x]$. Prove that α is an integer.

Exercise 4. Let $\mathbb{F}$ be a finite field of characteristic p. Prove that $|\mathbb{F}| = p^n$ for some positive integer n.

Exercise 5. Determine the minimal polynomial over $\mathbb{Q}$ for the element $1 + i$.

Exercise 6. Let $F = \mathbb{Q}(i)$. Prove that $x^3 - 2$ and $x^3 - 3$ are irreducible over F.

Exercise 7. Determine the degree of the extension $\mathbb{Q}(\sqrt{3 + 2\sqrt{2}})$ over $\mathbb{Q}$.

Exercise 8. Prove that if $[F(\alpha) : F]$ is odd then $F(\alpha) = F(\alpha^2)$.

Exercise 9. Let K/F be an algebraic extension and let R be a ring contained in K and containing F. Show that R is a subfield of K containing F.

Exercise 10. Suppose that F/K is a finite algebraic extension and $\alpha \in F$. Show that if A is the matrix of the linear transformation "multiplication by α" ($F \to F : x \mapsto \alpha x$) over K, then α is a root of the characteristic polynomial for A. This gives an effective procedure for determining an equation of degree n satisfied by an element α in an extension F of degree n. Use this procedure to obtain the monic polynomial of degree 3 satisfied by $\sqrt[3]{2}$ and by $1 + \sqrt[3]{2} + \sqrt[3]{4}$ (say over $\mathbb{Q}$).