Math 200b (Winter 2016) - Homework 3

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Exercise 1. Prove that the field $\mathbb{Q}(\alpha)$, where $\alpha^3 = 2$, has no automorphisms except the identity.

Exercise 2. Let $\alpha = 1 + \sqrt{2}$. Prove that there exists two embeddings of $\mathbb{Q}(\alpha)$ into $\mathbb{C}$, namely those sending α to $1 + \sqrt{2}$ and $1 - \sqrt{2}$ respectively.

Exercise 3. Let $\alpha = \sqrt{-5}$, $\beta = \sqrt{2}$. What is the degree of $\mathbb{Q}(\alpha, \beta)$ over $\mathbb{Q}$? Find an element γ such that $\mathbb{Q}(\alpha, \beta) = \mathbb{Q}(\gamma)$.

Exercise 4. Let $f(t)$ be an irreducible polynomial over a field F. Prove that the number of distinct roots of $f(t)$ in the algebraic closure $\overline{F}$ divides $\deg f(t)$.

Exercise 5. Let F be a field of characteristic $p \neq 0$. Prove that the field of rational functions $F(t)$ is not separable over its subfield $F(t^p)$.

Exercise 6. Let $F = \mathbb{Z}/p\mathbb{Z}$ be the field with p elements (p prime) and let $a \in F^{\times}$. Show that $f(t) = t^p - t - a$ is irreducible in $F[t]$.
Hint 1: Prove that if α is a root, then $\alpha + 1$ is a root as well.
Hint 2: Let $f = gh$ for some $g, h \in F[t]$ with $1 \leq \deg g < \deg f$. Write $g(t) = \gamma_0 + \gamma_1 t + \cdots + \gamma_{k-1} t^{k-1} + t^k$ and look at γ_{k-1}.