Math 200b (Winter 2016) - Homework 6

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Let F be a field and denote by $(\frac{a,b}{F})$ the quaternion algebra associated to $a, b \in F^\times$. Recall that $(\frac{a,b}{F})$ is the (noncommutative) F-algebra generated by two elements i and j satisfying $i^2 = a$, $j^2 = b$ and $ij = -ji$; a basis of $(\frac{a,b}{F})$ over F is thus given by $\{1, i, j, ij\}$. In what follows, we identify F with its image $F.1$ in $(\frac{a,b}{F})$. If $x = a + bi + cj + dij \in (\frac{a,b}{F})$, we note $\overline{x} = a - bi - cj - dij$ the conjugate of x.

Exercise 1. Prove that the center of $(\frac{a,b}{F})$ is F.

Exercise 2. Show that the trace $\operatorname{tr} x = x + \overline{x}$ and the norm $N(x) = x\overline{x}$ of $x \in (\frac{a,b}{F})$ both belong to F. Prove that $x^2 - \operatorname{tr}(x)x + N(x) = 0$ holds for all $x \in (\frac{a,b}{F})$.

Exercise 3. Prove that the algebra $(\frac{a,b}{F})$ does not contain any non-trivial two-sided ideals.

Exercise 4. Suppose F has an automorphism σ of order n. Let $D = F((t;\sigma))$ be the division algebra of skew Laurent series. (Recall, D has as underlying additive group the usual Laurent series in t, but with multiplication rule $ta = \sigma(a)t$ for $a \in F$.) Prove that $Z = F^\sigma((t^n;\sigma))$ is the center of D (here F^σ denotes the fixed field of σ). Find the dimension of D over Z.