# HOMEWORK 1

DUE 8 APRIL 2016

**1.** Let $\mathcal{C}$ be a category, and let $U_1$ and $U_2$ be objects in $\mathcal{C}$. Suppose $U_1$ and $U_2$ are both universally attracting. Show that there is a unique isomorphism $i : U_1 \longrightarrow U_2$. (For future reference, the same is true if they're both universally repelling, with the same proof.)

**2.** Remember that by "ring" we mean "ring with 1." Let $\mathcal{R}$ be the category of rings. If $R_1$ and $R_2$ are rings, then let $R_1 \times R_2$ be their set-theoretic product, which can also be given the natural structure of a ring.

(a) Show that $R_1 \times R_2$ is the product of $R_1$ and $R_2$ in $\mathcal{R}$.

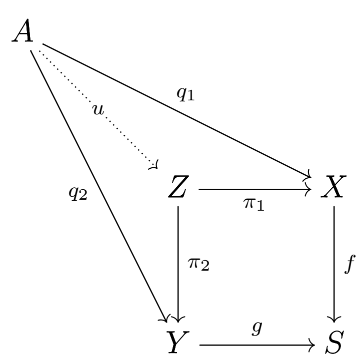
(b) Show that $R_1 \times R_2 \simeq R_1 \oplus R_2$ is not the coproduct of $R_1$ and $R_2$ in $\mathcal{R}$.

Note: We'll see later that coproducts do exist in the category $\mathcal{R}_{\text{comm}}$ of commutative rings; they're called tensor products.

**3.** Let $\mathcal{C}$ be a category. Let $X, Y, S$ be objects in $\mathcal{C}$ and $f : X \longrightarrow S, g : Y \longrightarrow S$ be morphisms in $\mathcal{C}$. A *fiber product* of $f$ and $g$ in $\mathcal{C}$ (or by abuse of terminology, fiber product of $X$ and $Y$ over $S$) is an object $Z$ in $\mathcal{C}$ together with morphisms $\pi_1 : Z \longrightarrow X$ and $\pi_2 : Z \longrightarrow Y$ such that

(i) $g \circ \pi_2 = f \circ \pi_1$;

(ii) for any object $A$ in $\mathcal{C}$ and any morphisms $q_1 : A \longrightarrow X, q_2 : A \longrightarrow Y$ such that $g \circ q_2 = f \circ q_1$ there exists a unique morphism $u : A \longrightarrow Z$ such that the diagram




is commutative.

Show that, if it exists, the fiber product $Z$ of $f$ and $g$ is unique up to isomorphism. (The fiber product is denoted $X \times_S Y$.)

**4.** Let $B$ be an abelian group. Let $F_B$ be the functor from the category of abelian groups to itself defined for an abelian group $A$ by

$$F_B(A) = \mathrm{Hom}(B, A) = \{f : B \longrightarrow A; f \text{ is a group homomorphism}\}.$$

(a) Show that $F_B$ is a covariant functor.
(b) Show that $F_B$ is left exact.
(c) Find a nontrivial abelian group $B$ such that $F_B$ is exact.
(d) Is $F_B$ always exact? Prove or find a counterexample.

**5.** Let $G$ be a group. Denote $\mathbb{Z}[G]$ the free abelian group (or free $\mathbb{Z}$-module) on the set $G$. That is,

$$\mathbb{Z}[G] = \left\{ \sum_{\sigma \in G} a_\sigma \sigma \,;\, a_\sigma \in \mathbb{Z}\, \forall \sigma \in G \text{ and all but finitely many } a_\sigma\text{'s are equal to zero} \right\}$$

with the natural addition.

(a) Show that $\mathbb{Z}[G]$ becomes a ring with the multiplication

$$\left( \sum_{\sigma \in G} a_\sigma \sigma \right) \cdot \left( \sum_{\tau \in G} b_\tau \tau \right) = \sum_{\sigma \in G} \left( \sum_{\sigma' \tau = \sigma} a_{\sigma'} b_\tau \right) \sigma.$$

(Do show that the multiplication is well-defined.)

(b) What is the unit in this ring?

(c) Show that the set $R$ of finitely supported functions $f : G \longrightarrow \mathbb{Z}$ becomes a ring under the usual function addition and multiplication given by convolution. That is, the elements of $R$ are maps of sets $f : G \longrightarrow \mathbb{Z}$ with the property that $f(\sigma) = 0$ for all but finitely many $\sigma \in G$; the addition is given by $(f+g)(\sigma) = f(\sigma) + g(\sigma)$ for all $\sigma \in G$; and the multiplication is given by

$$(f * g)(\sigma) = \sum_{\tau \in G} f(\tau) g(\tau^{-1}\sigma).$$

(d) Show that $\mathbb{Z}[G]$ is naturally isomorphic to $R$ (as rings).