Algebra qualifying exam practice

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Problem 1. Show that there is no simple group of order 56. (If you have time to waste, show that every simple group of order less than 59 is abelian. If you still have time to waste after this, try to show that there is no simple group of order 2016. This last part may be extremely tricky.)

Problem 2. Show that, up to isomorphism, there is only one simple group of order 60. Deduce that there is only one non-solvable group of order 60 and that $A_5 \cong \mathrm{SL}_2(\mathbb{F}_4)$. (This problem may be difficult.)

Problem 3. Let p be a prime and G a group of order $p(p+1)$. Show that G can not be simple.

Problem 4. Recall that the Frattini subgroup $\Phi(G)$ of a group G is the intersection of all the maximal subgroups of G. Show that $\Phi(G)$ is nilpotent when G is finite.

Problem 5. An element g in a group G is called superfluous when the following property holds: every time a subset X of G generates G, then $X - \{g\}$ also generates G. Show that the Frattini subgroup $\Phi(G)$ is precisely the set of superfluous elements.

Problem 6. Let p be a prime number and $a \in \mathbb{Q}$. Find the Galois group of the splitting field K of $t^p - a$ over $\mathbb{Q}$ in terms of p and a.

Problem 7. Let K be the splitting field of the polynomial $t^6 - 9t^4 + 6t^2 + 3$ over $\mathbb{Q}$. Is the Galois group of K over $\mathbb{Q}$ solvable?

Problem 8. Let $F \subset E$ be a field extension and K, L intermediate fields such that K is Galois over F and $K \cap L = F$. If $\alpha \in L$ is algebraic over F, show that the minimal polynomial of α over F is irreducible over K.

Problem 9. Let K be a purely inseparable extension of F. Show that $\mathrm{Aut}(K/F) = \{1\}$.

Problem 10. Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ are isomorphic $\mathbb{Q}$-modules. Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \times \mathbb{C}$ are isomorphic rings.

Problem 11. Show that the set of zero-divisors D of a commutative ring R is the union of prime ideals of R.

Problem 12. Let R be a ring and $e, f \in R$ such that $e^2 = e$, $f^2 = f$, $ef = fe = 0$ and $e + f = 1$. Show that eR is a projective right R-module.

Problem 13. Recall that a commutative ring R is said to be reduced if R has no nonzero nilpotent elements. Which ones of the following properties are local: R is reduced, R is an integral domain, R is a principal ideal domain.

Problem 14. Suppose that $f_1, \ldots, f_n$ generate the unit ideal in some commutative ring R. Show that if R_{f_i} is Noetherian for $1 \leq i \leq n$, then R is Noetherian.

Problem 15. Let $R \subset S$ be commutative rings with S integral over R. Show that $J(R) = J(S) \cap R$ (where J denotes the Jacobson radical).
Hint: show that if $x \in R \cap S^\times$, then $x \in R^\times$.