THIRD HOMEWORK, DUE FEBRUARY 1ST

1. Let M, N and P be R-modules and let F be a free R-module of rank n. Show that there are isomorphisms, where all but the last is natural:

(a)
$$M \underset{R}{\otimes} N \simeq N \underset{R}{\otimes} M.$$

(b)
$$M \underset{R}{\otimes} (N \underset{R}{\otimes} P) \simeq (M \underset{R}{\otimes} N) \underset{R}{\otimes} P.$$

(c)
$$R \underset{R}{\otimes} M \simeq M$$

(d)
$$M \underset{R}{\otimes} (N \oplus P) \simeq (M \underset{R}{\otimes} N) \oplus (M \underset{R}{\otimes} P).$$

(e)
$$F \underset{R}{\otimes} M \simeq M^n,$$

the direct sum of copies of M with itself n times.

2. Let m and n be integers. Identify $\mathbb{Z}_m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_n$.

3. Show that if M and N are two finitely generated (respectively Noetherian) R-modules over a ring R (respectively Noetherian), then so is $M \underset{R}{\otimes} N$.

4. Show that $\mathbb{C} \underset{\mathbb{R}}{\otimes} \mathbb{C}$ and $\mathbb{C} \underset{\mathbb{C}}{\otimes} \mathbb{C}$ are not isomorphic $\mathbb{R}$-modules.

5. Show that $\mathbb{Q} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ and $\mathbb{Q} \underset{\mathbb{Q}}{\otimes} \mathbb{Q}$ are isomorphic $\mathbb{Q}$-modules.

6. Show that
$$\mathbb{Q}/\mathbb{Z} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}/\mathbb{Z} \simeq 0.$$

Challenge Problem: 7. Show that tensor products commute with arbitrary direct sums but not with arbitrary direct products.