FOURTH HOMEWORK, DUE FEBRUARY 8TH

1. Let M, N and P be R-modules over a ring R. Show that there are natural isomorphisms:
(i)
$$\bigwedge^d (M \oplus N) \simeq \bigoplus_{i+j=d} \left(\bigwedge^i M \underset{R}{\otimes} \bigwedge^j N) \right).$$
(ii)
$$\mathrm{Hom}_R(M \underset{R}{\otimes} N, P) \simeq \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)).$$
2. Let V and W be vector spaces over a field F. Let
$$V^* = \mathrm{Hom}_F(V, F),$$
be the dual vector space. Show that there is a natural isomorphism
$$\mathrm{Hom}_F(V, W) \simeq V^* \underset{F}{\otimes} W.$$
3. Suppose that
$$M \longrightarrow N \longrightarrow P \longrightarrow 0,$$
is a sequence of R-modules.
Show that
$$0 \longrightarrow \mathrm{Hom}_R(P, Q) \longrightarrow \mathrm{Hom}_R(N, Q) \longrightarrow \mathrm{Hom}_R(M, Q),$$
is (left) exact for all R-modules Q if and only if the first sequence is (right) exact.
4. Suppose that
$$0 \longrightarrow M \longrightarrow N \longrightarrow P,$$
is a sequence of R-modules.
Show that
$$0 \longrightarrow \mathrm{Hom}_R(Q, M) \longrightarrow \mathrm{Hom}_R(Q, N) \longrightarrow \mathrm{Hom}_R(Q, P),$$
is left exact for all R-modules Q if and only if the first sequence is left exact.
5. Suppose that
$$M \longrightarrow N \longrightarrow P \longrightarrow 0,$$
is a right exact sequence of R-modules.
Show that
$$M \underset{R}{\otimes} Q \longrightarrow N \underset{R}{\otimes} Q \longrightarrow P \underset{R}{\otimes} Q \longrightarrow 0.$$
is right exact for all R-modules Q.

6. Give examples to show that one cannot extend (3–5) to short exact sequences. For example, even if

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0,$$

is a short exact sequence of R-modules then

$$0 \longrightarrow \mathrm{Hom}_R(M, Q) \longrightarrow \mathrm{Hom}_R(N, Q) \longrightarrow \mathrm{Hom}_R(M, Q) \longrightarrow 0,$$

is not necessarily a short exact sequence.

Challenge Problem: 7. Give an example of a PID R and a matrix A with entries in R such that one cannot realise the gcd d of A as an entry of A by elementary row and column operations.