HOMEWORK 3

DUE 28 APRIL 2017

1. Prove that $\varinjlim \frac{1}{p^i}\mathbb{Z} = \mathbb{Z}\left[\frac{1}{p}\right]$.

2. Atiyah-MacDonald, Chapter 2, Exercise 20.

3. Let p be a prime number. For $n \geq m$ let $f_{nm} : \mathbb{Z}/p^n\mathbb{Z} \longrightarrow \mathbb{Z}/p^m\mathbb{Z}$ be the canonical projection, i.e. $f_{nm}(a \bmod p^n) = a \bmod p^m$.

(a) Show that $\{\mathbb{Z}/p^n\mathbb{Z}\}$ with homomorphisms f_{nm} forms an inverse system of commutative rings. Let $\mathbb{Z}_p$ denote $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$

(b) Find the canonical image of $\mathbb{Z}$ in $\mathbb{Z}_p$ and show that $\mathbb{Z}_p$ is an integral domain.

(c) Show that $\mathbb{Z}_p$ is a local ring and an principal ideal domain.

The ring $\mathbb{Z}_p$ is called the ring of p-adic integers.

4. Let p be a prime and let R be the set of formal power series in p :

$$R = \left\{ \sum_{n=0}^{\infty} a_n p^n \,;\, a_n = 0, 1, \ldots, p-1 \right\}.$$

(a) Show that R is a commutative ring under the addition and multiplication of power series (do show that multiplication makes sense!).

(b) Show that $\mathbb{Z}_p$ is naturally isomorphic to R.

5. Let $\mathbb{N}$ be the set of positive integers ordered by divisibility. Observe that

$$\{\mathbb{Z}/n\mathbb{Z}\}_{n\in\mathbb{N}}$$

forms an inverse system of commutative rings with the canonical homomorphisms $\mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z}$ for $m \mid n$. Let $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$. Show that

$$\hat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p.$$

6. Let p be a prime and let $\mathbb{F}_p$ be a field with p elements.

(a) Show that $\mathbb{F}_p$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. (Use the shortest proof possible…)

(b) Let $K/\mathbb{F}_p$ be a finite extension. Then K is a finite dimensional vector space over $\mathbb{F}_p$ and hence has $q = p^n$ elements for some n. Show that $K/\mathbb{F}_p$ is Galois with cyclic Galois group generated by

$$\phi : K \longrightarrow K, \quad x \mapsto x^p.$$

As you probably already know, ϕ is called the p-th power Frobenius.

(c) Show that for each $n \geq 1$, there is exactly one field K with $\mathbb{F}_p \subseteq K \subseteq \bar{\mathbb{F}}_p$ of degree n over $\mathbb{F}_p$.

(d) Show that $\mathrm{Gal}\left(\bar{\mathbb{F}}_p/\mathbb{F}_p\right) \cong \hat{\mathbb{Z}}$. (It is also true that $\mathrm{Gal}\left(\bar{\mathbb{F}}_q/\mathbb{F}_q\right) \cong \hat{\mathbb{Z}}$ with the same proof.)

7. Let p be a prime number. For any integer $a \neq 0$ we denote $v_p(a) = n \iff p^n$ is the highest power of p dividing a. (Sometimes this is extended to include $v_p(0) = \infty$.)

(a) For $a \in \mathbb{Z}$, define the *p-adic absolute value* of a to be $|a|_p = \dfrac{1}{p^{v_p(a)}}$ if $a \neq 0$ and $|0|_p = 0$. Show that

$$|a + b|_p \leq \max\{|a|_p, |b|_p\}$$

for all $a, b \in \mathbb{Z}$. Under what conditions is this an equality?

(b) Show that

$$(a, b) \mapsto |a - b|_p, \quad a, b \in \mathbb{Z},$$

defines a metric on $\mathbb{Z}$ which makes $\mathbb{Z}$ into a topological group (that is, with respect to this topology, addition $+ : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ is continuous).

(c) Extend the absolute value $|\cdot|_p$ from $\mathbb{Z}$ to one on the p-adic integers $\mathbb{Z}_p$ in a natural way so that, in the topology on $\mathbb{Z}_p$ every Cauchy sequence in $\mathbb{Z}$ converges in $\mathbb{Z}_p$.

(d) Let $\{a_i\}_{i=1}^{\infty}$ be an infinite sequence in $\mathbb{Z}_p$. Under which conditions will the series $\sum a_i$ converge?