

# Homework 6

Tuesday, February 20, 2018

9:54 PM

**1** Let  $R$  be a local unital commutative ring, and  $\text{Max}(R) = \{\mathfrak{m}\}$ .

(a) Let  $M$  be a finitely generated  $R$ -module. Suppose  $M = \mathfrak{m}M$ .

Prove that  $M = 0$ . (Hint. Let  $x_1, \dots, x_d$  be a generating set of  $M$ . By assumption,  $\exists a_{ij} \in \mathfrak{m}$ ,  $x_i = \sum_{j=1}^d a_{ij} x_j$ . Hence  $(I - [a_{ij}]) \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix} = \vec{0}$ . Show that  $I - [a_{ij}] \in GL_d(R)$ ; and deduce  $x_i = 0$ ; and so  $M = 0$ )

(b) Let  $M$  be a finitely generated  $R$ -module. Let  $d(M)$  be the minimum number of generators of  $M$ . Prove that

$$d(M) = \dim_{(R/\mathfrak{m})} (R/\mathfrak{m}) \otimes_R M.$$

(c) Let  $M$  be a finitely generated projective  $R$ -module. Prove that

$M$  is free. (Hint. Let  $d(M) = d$ . Then  $0 \rightarrow N \rightarrow R^d \rightarrow M \rightarrow 0$

splits; and so  $R^d \simeq M \oplus N \stackrel{?}{\Rightarrow} (R/\mathfrak{m}) \otimes_R R^d \simeq ((R/\mathfrak{m}) \otimes_R M) \oplus ((R/\mathfrak{m}) \otimes_R N) \stackrel{?}{\Rightarrow} N = 0$ .)

**2** Let  $R$  be a unital commutative ring.

(a) Let  $S$  be a multiplicatively closed subset of  $R$ . Prove that

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$$S^{-1}R \otimes_R M \simeq S^{-1}M \text{ as } S^{-1}R\text{-modules.}$$

(b) Suppose  $R_1$  and  $R_2$  are unital commutative rings, and

$\phi: R_1 \rightarrow R_2$  is a ring homomorphism. Prove that if  $M$  is a

flat  $R_1$ -module, then  $R_2 \otimes_{R_1} M$  is a flat  $R_2$ -module.

(c) Prove that, if  $M$  is a flat  $R$ -mod, then  $S^{-1}M$  is a flat  $S^{-1}R$ -module; in particular if  $M$  is a flat  $R$ -module, then  $\forall \mathfrak{p} \in \text{Spec}(R)$ ,  $M_{\mathfrak{p}}$  is a flat  $R_{\mathfrak{p}}$ -module.

$$(d) \text{ Prove } S^{-1}(M_1 \otimes_R M_2) \simeq S^{-1}M_1 \otimes_{S^{-1}R} S^{-1}M_2, \frac{x_1 \otimes x_2}{1} \mapsto \frac{x_1}{1} \otimes \frac{x_2}{1}.$$

(e) Prove that, if  $M_{\mathfrak{p}}$  is a flat  $R_{\mathfrak{p}}$ -module for any  $\mathfrak{p} \in \text{Spec}(R)$ ,

then  $M$  is flat.

(Hint. Suppose  $0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$  is S.E.S.. Show

$$0 \rightarrow M_{\mathfrak{p}} \otimes (N_1)_{\mathfrak{p}} \rightarrow M_{\mathfrak{p}} \otimes (N_2)_{\mathfrak{p}} \rightarrow M_{\mathfrak{p}} \otimes (N_3)_{\mathfrak{p}} \rightarrow 0 \text{ is S.E.S. ;}$$

$$\text{deduce } 0 \rightarrow (M \otimes N_1)_{\mathfrak{p}} \rightarrow (M \otimes N_2)_{\mathfrak{p}} \rightarrow (M \otimes N_3)_{\mathfrak{p}} \rightarrow 0 \text{ is S.E.S.}$$

Use HW4, Problem 1(c) and (d).)

**3.** Suppose  $R$  is a unital commutative local ring. Suppose  $M$  and  $N$  are two finitely generated  $R$ -modules and  $M \otimes_R N = 0$ . Prove that

either  $M=0$  or  $N=0$  (Hint. Use an idea similar to **1.**(b).)

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▣ Suppose  $R$  is a unital ring and  $0 \rightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \rightarrow 0$

is a short exact sequence of right  $R$ -modules. Suppose  $M_3$

is flat. Prove, for any left  $R$ -mod.  $N$ ,

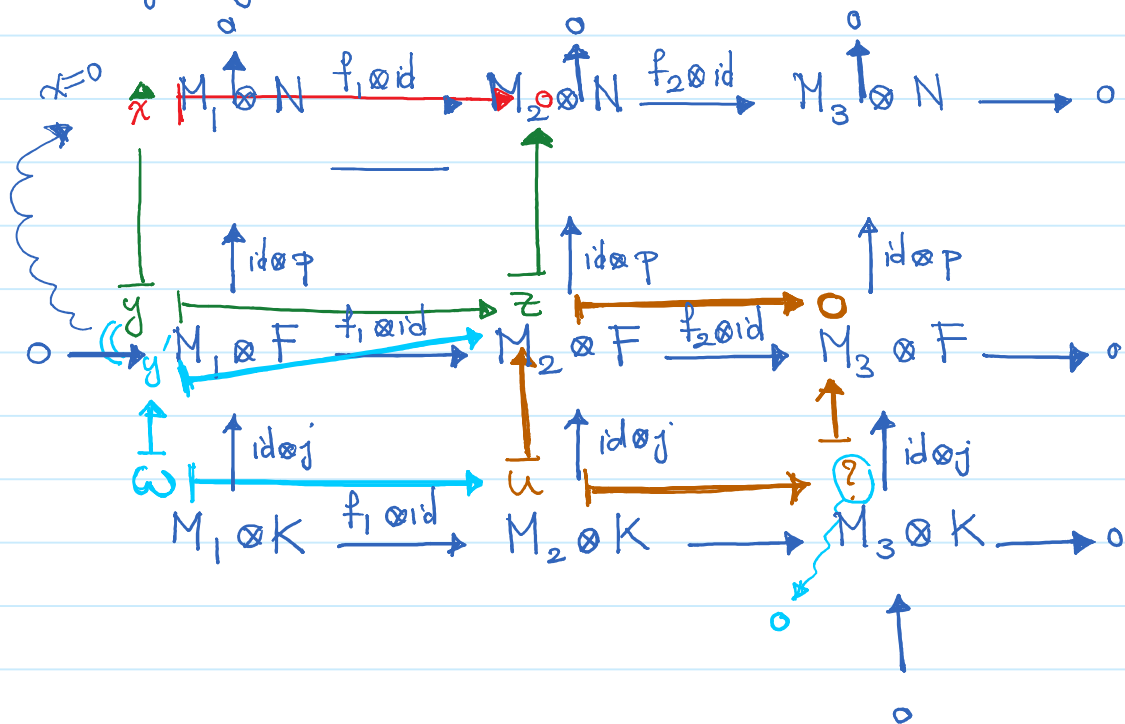
$$0 \rightarrow M_1 \otimes_R N \xrightarrow{f_1 \otimes \text{id}} M_2 \otimes_R N \xrightarrow{f_2 \otimes \text{id}} M_3 \otimes_R N \rightarrow 0$$

is a S.E.S.

(Hint. Notice that there is a S.E.S.  $0 \rightarrow K \xrightarrow{j} F \xrightarrow{p} N \rightarrow 0$

where  $F$  is free. Then show that we get the following

commuting diagram where all the rows and columns are exact:



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[5] Suppose  $R$  is a unital ring and  $0 \rightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \rightarrow 0$  is a S.E.S. of right  $R$ -modules. Suppose  $M_3$  is flat. Prove that  $M_1$  is flat if and only if  $M_2$  is flat.

(Hint. Use [4].)

[6] Suppose  $D$  is an integral domain, and  $M$  is a  $D$ -module.

(a) Prove Free  $\Rightarrow$  Projective  $\Rightarrow$  flat  $\Rightarrow$  torsion-free.

(proved in class)

(proved in class)

prove only this part

(b) If  $D$  is a PID and  $M$  is f.g., then all the above properties are equivalent.

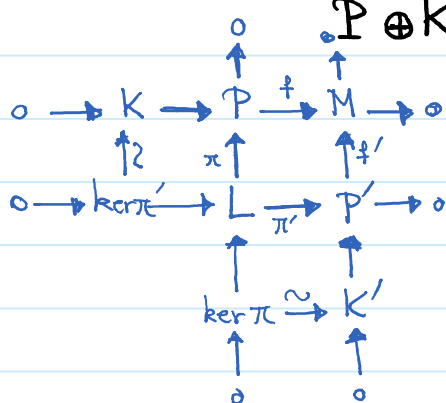
(c) Prove that  $\mathbb{Q}$  is NOT a projective  $\mathbb{Z}$ -module, but it is a flat  $\mathbb{Z}$ -module.

[7]. Suppose  $0 \rightarrow K \rightarrow P \xrightarrow{f} M \rightarrow 0$  and  $0 \rightarrow K' \rightarrow P' \xrightarrow{f'} M \rightarrow 0$

are S.E.S. Suppose  $P$  and  $P'$  are projective. Prove

$$P \oplus K' \cong P' \oplus K.$$

(Hint:



$$L := \{(\alpha, \alpha') \in P \oplus P' \mid f(\alpha) = f'(\alpha')\}.$$

(this is called the fiber product of

