# Homework 6

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$\boxed{1}$ Let $R$ be a local unital commutative ring, and $\mathrm{Max}(R)=\{\mathfrak{m}\}$.

(a) Let $M$ be a finitely generated $R$-module. Suppose $M=\mathfrak{m}M$.

Prove that $M=0$. (Hint. Let $x_1,\ldots,x_d$ be a generating set of $M$. By assumption, $\exists\, a_{ij}\in\mathfrak{m}$, $x_i=\sum_{j=1}^{d} a_{ij}x_j$. Hence $\left(I-[a_{ij}]\right)\begin{bmatrix}x_1\\ \vdots\\ x_d\end{bmatrix}=\vec{0}$. Show that $I-[a_{ij}]\in GL_d(R)$; and deduce $x_i=0$; and so $M=0$)

(b) Let $M$ be a finitely generated $R$-module. Let $d(M)$ be the minimum number of generators of $M$. Prove that

$$d(M)=\dim_{(R/\mathfrak{m})} R/\mathfrak{m}\otimes_R M.$$

(c) Let $M$ be a finitely generated projective $R$-module. Prove that $M$ is free. (Hint. Let $d(M)=d$. Then $0\to N\to R^d\to M\to 0$ splits; and so $R^d\simeq M\oplus N \overset{?}{\Rightarrow} (R/\mathfrak{m})\otimes_R R^d\simeq\left((R/\mathfrak{m})\otimes_R M\right)\oplus\left(R/\mathfrak{m}\otimes_R N\right)$ $\overset{?}{\Rightarrow} N=0$.)

$\boxed{2}$ Let $R$ be a unital commutative ring.

(a) Let $S$ be a multiplicatively closed subset of $R$. Prove that

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$S^{-1}R \otimes_R M \simeq S^{-1}M$ as $S^{-1}R$- modules.

(b) Suppose $R_1$ and $R_2$ are unital commutative rings, and $\phi: R_1 \to R_2$ is a ring homomorphism. Prove that if $M$ is a flat $R_1$-module, then $R_2 \otimes_{R_1} M$ is a flat $R_2$-module.

(c) Prove that, if $M$ is a flat $R$-mod, then $S^{-1}M$ is a flat $S^{-1}R$-module; in particular if $M$ is a flat $R$-module, then $\forall \mathfrak{p} \in \operatorname{Spec}(R)$, $M_{\mathfrak{p}}$ is a flat $R_{\mathfrak{p}}$- module.

(d) Prove $S^{-1}(M_1 \otimes_R M_2) \simeq S^{-1}M_1 \otimes_{S^{-1}R} S^{-1}M_2$, $\frac{x_1 \otimes x_2}{1} \mapsto \frac{x_1}{1} \otimes \frac{x_2}{1}$.

(e) Prove that, if $M_{\mathfrak{p}}$ is a flat $R_{\mathfrak{p}}$-module for any $\mathfrak{p} \in \operatorname{Spec}(R)$, then $M$ is flat.

(Hint. Suppose $0 \to N_1 \to N_2 \to N_3 \to 0$ is S.E.S.. Show

$0 \to M_{\mathfrak{p}} \otimes (N_1)_{\mathfrak{p}} \to M_{\mathfrak{p}} \otimes (N_2)_{\mathfrak{p}} \to M_{\mathfrak{p}} \otimes (N_3)_{\mathfrak{p}} \to 0$ is S.E.S.;

deduce $0 \to (M \otimes N_1)_{\mathfrak{p}} \to (M \otimes N_2)_{\mathfrak{p}} \to (M \otimes N_3)_{\mathfrak{p}} \to 0$ is S.E.S.

Use HW4, Problem 1(c) and (d).)

$\boxed{3.}$ Suppose $R$ is a unital commutative local ring. Suppose $M$ and $N$ are two finitely generated $R$-modules and $M \otimes_R N = 0$. Prove that either $M=0$ or $N=0$ (Hint. Use an idea similar to $\boxed{1}$.(b).)

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$\boxed{4}$ Suppose $R$ is a unital ring and $0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$ is a short exact sequence of right $R$-modules. Suppose $M_3$ is <u>flat</u>. Prove, for any left $R$-mod. $N$,

$$0 \to M_1 \otimes_R N \xrightarrow{f_1 \otimes id} M_2 \otimes_R N \xrightarrow{f_2 \otimes id} M_3 \otimes_R N \to 0$$

is a S.E.S.

(<u>Hint</u>. Notice that there is a S.E.S. $0 \to K \xrightarrow{j} F \xrightarrow{p} N \to 0$ where $F$ is free. Then show that we get the following commuting diagram where all the rows and columns are exact:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
 & M_1 \otimes N \arrow[u] \arrow[r, "f_1 \otimes id"] & M_2 \otimes N \arrow[u] \arrow[r, "f_2 \otimes id"] & M_3 \otimes N \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & M_1 \otimes F \arrow[u, "id \otimes p"] \arrow[r, "f_1 \otimes id"] & M_2 \otimes F \arrow[u, "id \otimes p"] \arrow[r, "f_2 \otimes id"] & M_3 \otimes F \arrow[u, "id \otimes p"] \arrow[r] & 0 \\
 & M_1 \otimes K \arrow[u, "id \otimes j"] \arrow[r, "f_1 \otimes id"] & M_2 \otimes K \arrow[u, "id \otimes j"] \arrow[r] & M_3 \otimes K \arrow[u, "id \otimes j"] \arrow[r] & 0 \\
 & & & 0 \arrow[u] &
\end{tikzcd}$$

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$\boxed{5}$ Suppose $R$ is a unital ring and $0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$ is a S.E.S. of right $R$-modules. Suppose $M_3$ is flat. Prove that $M_1$ is flat if and only if $M_2$ is flat.

(Hint. Use $\boxed{4}$.)

$\boxed{6}$ Suppose $D$ is an integral domain, and $M$ is a $D$-module.

(a) Prove Free $\Rightarrow$ Projective $\Rightarrow$ flat $\Rightarrow$ torsion-free.

(proved in class) (proved in class) prove only this part

(b) If $D$ is a PID and $M$ is f.g., then all the above properties are equivalent.

(c) Prove that $\mathbb{Q}$ is NOT a projective $\mathbb{Z}$-module, but it is a flat $\mathbb{Z}$-module.

$\boxed{7}$. Suppose $0 \to K \to P \xrightarrow{f} M \to 0$ and $0 \to K' \to P' \xrightarrow{f'} M \to 0$ are S.E.S. Suppose $P$ and $P'$ are projective. Prove

$$P \oplus K' \simeq P' \oplus K.$$

(Hint.

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & K \arrow[r] & P \arrow[r, "f"] \arrow[u] & M \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \ker\pi' \arrow[r] \arrow[u, "\wr"] & L \arrow[r, "\pi'"] \arrow[u, "\pi"] & P' \arrow[r] \arrow[u, "f'"] & 0 \\
 & & \ker\pi \arrow[r, "\sim"] \arrow[u] & K' \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

$L := \{(x, x') \in P \oplus P' \mid f(x) = f'(x')\}$.)

(this is called the fiber product of

$$\begin{tikzcd}
 & L \arrow[dl] \arrow[dr] & \\
P \arrow[dr, "f"'] & & P' \arrow[dl, "f'"] \\
 & M &
\end{tikzcd}$$

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