Homework 7

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1. (a) Let F be a field of characteristic not 2. Let $a, b \in F^{\times} \setminus F^{\times 2}$.

   Prove that $[F[\sqrt{a}, \sqrt{b}] : F] = 4$ if and only if $ab \notin F^{\times 2}$.

   And, if $ab \in F^{\times 2}$, then $F[\sqrt{a}, \sqrt{b}] = F[\sqrt{a}]$.

   (b) Prove that $\sqrt[3]{2} \notin \mathbb{Q}[\sqrt{a_1}, \sqrt{a_2}, \cdots, \sqrt{a_n}]$ for some $a_i \in \mathbb{Q}^{\times}$.

2. Let A be an F-algebra. Suppose $\dim_F A < \infty$ and A is an integral domain. Prove that A is a field.

   (Hint. Consider $l_a : A \to A$, $l_a(a') := aa'$ as an F-linear map.)

3. Let K/F be a field extension. Suppose $F \subseteq K_1 \subseteq K$ and $F \subseteq K_2 \subseteq K$ are subfields. And let $K_1 K_2$ be the subfield of K that is generated by $K_1 \cup K_2$. (Smallest subfield that contains both K_1 and K_2).

(a) Suppose $[K_1 : F] < \infty$ and $[K_2 : F] < \infty$. Prove that

$$K_1 K_2 = \left\{ \sum_{i=1}^{m} a_i b_i \;\middle|\; a_1, \ldots, a_m \in K_1,\ b_1, \ldots, b_m \in K_2 \right\}.$$

(b) Suppose $[K_i : F] < \infty$ for $i = 1, 2$. Prove that there is an onto F-algebra homomorphism $\phi : K_1 \otimes_F K_2 \to K_1 K_2$, $\phi(a \otimes b) = ab$.

(c) In the setting of (b), prove that

$$K_1 \otimes_F K_2 \text{ is a field} \iff \phi \text{ is an isomorphism} \iff [K_1 K_2 : F] = [K_1 : F][K_2 : F].$$

(d) Prove $\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{3}) \simeq \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

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4. Let $E \subseteq \mathbb{C}$ be a splitting field of $x^p - 2$ over $\mathbb{Q}$ where p is an odd prime. (a) Show that $E = \mathbb{Q}[\zeta_p, \sqrt[p]{2}]$ where $\zeta_p = e^{\frac{2\pi i}{p}}$. (b) Prove that $[E:\mathbb{Q}] = p(p-1)$.

5. (a) Prove that $\mathbb{F}_{p^m} \subseteq \mathbb{F}_{p^n}$ if and only if $m \mid n$.

(Hint. $(\Rightarrow)$ Let $[\mathbb{F}_{p^n}:\mathbb{F}_{p^m}]=d$. Then $p^n = (p^m)^d$.
$(\Leftarrow)$ Show $x^{p^m} - x \mid x^{p^n} - x$.)

(b) Let $f(x) \in \mathbb{F}_p[x]$ be a monic irreducible polynomial of degree d. Prove that $f(x) \mid x^{p^d} - x$.

(Hint. $\exists$ $E/\mathbb{F}_p$ s.t. $E = \mathbb{F}_p[\alpha]$ and $f(x)$ is the minimal poly. of α.)

(c) Suppose $f(x) \in \mathbb{F}_p[x]$ is irreducible and $f(x) \mid x^{p^n} - x$. Prove that $\deg(f) \mid n$.

(Hint. Use part (a).)

(d) Let $P_d := \{ f(x) \in \mathbb{F}_p[x] \mid \deg f = d, \text{ irreducib., monic} \}$. Prove that

$$\prod_{d \mid n} \prod_{f(x) \in P_d} f(x) = x^{p^n} - x.$$

Deduce that $p^n = \sum_{d \mid n} d\,|P_d|$.

(Remark. Using Möbius inversion, one can deduce that $|P_n| = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) p^d$.)

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6. Prove that $x^p - x + a$ is irreducible in $\mathbb{F}_p[x]$ for any $a \in \mathbb{F}_p^\times$.

(Hint. Let E be a splitting field of $f(x) = x^p - x + a$, and $\alpha \in E$ be a zero of $f(x)$. Show that $f(x) = (x-\alpha)(x-\alpha-1)\cdots(x-\alpha-p+1)$. Suppose $\deg(m_\alpha) = d$. Looking at the coeff. of x^{d-1}, deduce that $m_\alpha = f$.)

7. Suppose $[F[\alpha]:F]$ is odd. Prove that $F[\alpha] = F[\alpha^2]$.

8. Let F be a field, and $f(x) \in F[x] \setminus F$. Suppose E is a splitting field of $f(x)$ over E.

(a) Prove that, if $\gcd(f, f') \neq 1$, then $F[x]/\langle f(x)\rangle \otimes_F E$ has a non-zero nilpotent element.

(b) Prove that, if $\gcd(f, f') = 1$, then

$$F[x]/\langle f(x)\rangle \otimes_F E \simeq \underbrace{E \oplus \cdots \oplus E}_{\deg f},$$

as F-algebras.