Determinants of Perturbations of Finite Toeplitz Matrices

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One of the purposes of this talk is to describe a Helton observation that fundamentally made the computing the asymptotics of determinants of finite matrices quite simple.

We begin with the classical Szegö Limit Theorem and Toeplitz matrices.

Toeplitz Matrices

The Strong Szegö Limit Theorem states that if the symbol ϕ defined on the unit circle has a sufficiently well-behaved logarithm then the determinant of the Toeplitz matrix

$$T_N(\phi) = (\phi_{j-k})_{j,k=0,\cdots,N-1}$$

where

$$\phi_k = \frac{1}{2\pi}\int_0^{2\pi} \phi(e^{i\theta})e^{-ik\theta}\ d\theta$$

has the asymptotic behavior

$$D_N(\phi) = \det T_N(\phi) \sim G(\phi)^N E(\phi) \quad \text{as } N \to \infty.$$

Here the constants are

$$G(\phi) = e^{(\log \phi)_0}$$

$$E(\phi) = \exp\left(\sum_{k=1}^{\infty} k\,(\log \phi)_k\,(\log \phi)_{-k}\right).$$

The last constant $E(\phi)$ can also be described by

$$\det\left(T(\phi)\,T(\phi^{-1})\right)$$

where

$$T(\phi) = (\phi_{j-k}) \quad 0 \leq j, k < \infty$$

is the Toeplitz Operator defined on the Hardy space.

This constant makes sense because if ϕ is sufficiently well-behaved, then the operator

$$T(\phi)T(\phi^{-1}) - I$$

is trace class.

Here is a sketch of the proof.

The finite Toeplitz matrix $T_N(\phi)$ can be thought of as the upper left hand corner of the matrix representation of the operator $T(\phi)$. We can think of it then as

$$P_N T(\phi) P_N$$

where

$$P_N : \{x_n\}_{n=0}^{\infty} \in \ell^2 \mapsto \{y_n\}_{n=0}^{\infty} \in \ell^2, \qquad y_n = \begin{cases} x_n & \text{if } n < N \\ 0 & \text{if } n \geq N \end{cases}.$$

Now suppose that U is an operator whose matrix representation has an upper triangular form. Then

$$P_N U P_N = U P_n.$$

If L is an operator whose matrix representation has an lower triangular form. Then

$$P_N L P_N = P_n L.$$

So if we had an operator of the form LU, then

$$P_N L U P_N = P_N L P_N U P_N$$

and the corresponding determinants would be easy to compute.

What happens for Toeplitz operators is the opposite.

If the symbol is sufficiently nice then $T(\phi) = UL$ so we need to do something else. We write

$$P_N T(\phi) P_N = P_N U L P_N$$

$$= P_N L L^{-1} U L U^{-1} U P_N = P_N L P_N L^{-1} U L U^{-1} P_N U P_N$$

Now what else makes this works is that it turns out that the operator $L^{-1} U L U^{-1}$ is actually $T(\phi) T(\phi^{-1})$ and we know that this is I plus a trace class operator and thus has a well defined infinite determinant.

So putting this all together we have that

$$\det P_N L P_N \times \det P_N U P_N = G(\phi)^N$$

and that

$$\lim_{N->\infty} P_N L^{-1} U L U^{-1} P_N = \det T(\phi) T(\phi^{-1}).$$

Random Matrix Ensembles

There is a fundamental connection between determinants of Toeplitz matrices and random matrix ensembles.

For example, one can consider the Circular Unitary Ensemble (CUE) with joint density a constant times

$$\prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^2.$$

A linear statistic for this ensemble is a random variable of the form

$$S_N = \sum_{j=1}^{N} f(e^{i\theta_j}),$$

and it is this quantity which is connected to a Toeplitz determinant.

More precisely, if we define $g(\lambda)$ to be

$$\frac{1}{(2\pi)^N N!} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{j=1}^{N} e^{i\lambda f(e^{i\theta_j})} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^2 d\theta_1 \ldots d\theta_N$$

then $g(\lambda)$ is identically equal to

$$\det \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\lambda f(\theta)} e^{-i(j-k)\theta} d\theta \right)_{j,k=0,\ldots,N-1}.$$

The last determinant is a Toeplitz determinant with symbol

$$\phi(\theta) = e^{i\lambda f(e^{i\theta})}.$$

The identity holds because a very old result due to Andréief (1883) says that

$$\frac{1}{N!}\int \cdots \int \det(f_j(x_k))\det(g_j(x_k))dx_1 \cdots dx_N$$
$$= \det\left(\int f_j(x)g_k(x)dx\right)_{j,k=1,\cdots,N}.$$

One is interested in g because it is the inverse Fourier transform of the density of the linear statistic.

In the opposite sense, the Toeplitz determinant can be thought of as an average or expectation with respect to CUE.

Asymptotics of the determinant gives us information about the linear statistic. This is especially useful when the function f is smooth enough, because we may appeal to the Strong Szegö Limit Theorem to tell us asymptotically the behavior of the density function.

To obtain asymptotic information about the linear statistic we apply the Strong Szegö Limit Theorem.

This shows

$$g(\lambda) \sim G(\phi)^N E(\phi), \quad \phi(e^{i\theta}) = e^{i\lambda f(e^{i\theta})}$$

where

$$G(\phi)^N = \exp\left(i\lambda \frac{N}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) d\theta \right)$$

and

$$E(\phi) = \exp\left(-\lambda^2 \sum_{k=1}^{\infty} k f_k f_{-k} \right).$$

We see that we can interpret the last formula as saying that asymptotically as $N \to \infty$: For a smooth function f the distribution of

$$S_N - N\mu$$

where

$$S_N = \sum_{j=1}^{N} f(e^{i\theta_j}), \quad \mu = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) d\theta$$

converges to a Gaussian distribution with mean zero and variance given by

$$\sigma^2 = \sum_{1}^{\infty} k f_k f_{-k} = \sum_{1}^{\infty} k |f_k|^2$$

(The last equality holds if f is real-valued.)

Other examples that arise from RMT

It has also known that for if one considers averages for $O^+(2N)$, then the corresponding determinant is of a finite Toeplitz plus Hankel matrix and is of the form

$$\det\left(a_{j-k} + a_{j+k}\right)_{j,k=0,\ldots,N-1}$$

where subscripts denote Fourier coefficients and the function a is assumed to be even.

Hence we are interested in the determinants of a sum of a finite Toeplitz plus a “certain type” of Hankel matrix.

To be a little more general we are going to consider a set of operators and associated spaces the we will call compatible pairs.

Let $\mathcal{S}$ stand for a unital Banach algebra of functions on the unit circle continuously embedded into $L^\infty(\mathbb{T})$ with the property that $a \in \mathcal{S}$ implies that $\tilde{a} \in \mathcal{S}$ and $Pa \in \mathcal{S}$.

Here $\tilde{a}(e^{i\theta}) = a(e^{-i\theta})$.

and P is the Riesz projection defined by

$$P : \sum_{k=-\infty}^{\infty} a_k e^{ik\theta} \to \sum_{k=0}^{\infty} a_k e^{ik\theta}.$$

Moreover, define

$$\begin{aligned} \mathcal{S}_- &= \Big\{ a \in \mathcal{S} \ : \ a_n = 0 \text{ for all } n > 0 \Big\}, \\ \mathcal{S}_0 &= \Big\{ a \in \mathcal{S} \ : \ a = \tilde{a} \Big\}. \end{aligned}$$

Assume that $M : a \in L^\infty \mapsto M(a) \in \mathcal{L}(\ell^2)$ is a continuous linear map such that:

(a) If $a \in \mathcal{S}$, then $M(a) - T(a) \in \mathcal{C}_1(\ell^2)$ and

$$\|M(a) - T(a)\|_{\mathcal{C}_1(\ell^2)} \leq C \|a\|_{\mathcal{S}}.$$

(b) If $a \in \mathcal{S}_-$, $b \in \mathcal{S}$, $c \in \mathcal{S}_0$, then

$$M(abc) = T(a)M(b)M(c).$$

(c) $M(1) = I$.

Then we say M and $\mathcal{S}$ are compatible pairs.

Concrete examples

All of the following can be realized as compatible pairs with an appropriate Banach algebra. We define the Hankel operator $H(a)$ with symbol a by its with matrix representation

$$H(a) = (a_{j+k+1}), \quad 0 \leq j, k < \infty.$$

(I) $M(a) = T(a) + H(a)$,
(II) $M(a) = T(a) - H(a)$,
(III) $M(a) = T(a) - H(t^{-1}a)$ with $t = e^{i\theta}$,
(IV) $M(a) = (T(a) + H(ta))R$ with $R = \mathrm{diag}(1/2, 1, 1, \dots)$.

The matrix representations of the operators are of the form

$$a_{j-k} \pm a_{j+k-\varkappa+1}$$

with $\varkappa = 0, 1, -1$.

For each of the previous four examples we can take the Banach algebra to be the Besov class. This is the class of all functions a defined on the unit circle for which

$$\int_{-\pi}^{\pi} \frac{1}{y^2} \int_{-\pi}^{\pi} |a(e^{ix+iy}) + a(e^{ix-iy}) - 2a(e^{ix})\,| dxdy < \infty.$$

A function a is in this class if and only if the Hankel operators $H(a)$ and $H(\tilde{a})$ are both trace class.

Moreover the Riesz projection is bounded on this class and an equivalent norm is given by

$$|a_0| + \|H(a)\|_{\mathcal{C}_1} + \|H(\tilde{a})\|_{\mathcal{C}_1}.$$

We are interested in the determinants (where the matrices or operators are always thought of as acting on the image of the projection of the appropriate space) of

$$P_N M(a) P_N.$$

Theorem

Let M and $\mathcal{S}$ be a compatible pair, and let $b \in \mathcal{S}$ and $a = \exp(b)$. Then

$$\det P_N M(a) P_N \sim G[a]^N \hat{E}[a] \quad \text{as } N \to \infty,$$

where

$$\hat{E}[a] = \exp\left(\mathrm{trace}(M(b) - T(b)) - \frac{1}{2}\mathrm{trace}\, H(b)^2 + \mathrm{trace}\, H(b) H(\tilde{b})\right).$$

Borodin-Okunkov/Case-Geronimo Identity

We can also produce an exact identity for such matrices. The following is for even functions, but can be made more general.

Let M and $\mathcal{S}$ be a compatible pair, and let $b_+ \in \mathcal{S}_+$. Put $a = a_+\tilde{a}_+ = \exp(b)$ with $a_+ = \exp(b_+)$, $b = b_+ + \tilde{b}_+$. Then

$$\det P_N M(a) P_N = G[a]^N \hat{E}[a] \det(I + Q_N K Q_N),$$

where

$$\hat{E}[a] = \exp\left(\mathrm{trace}(M(b) - T(b)) + \frac{1}{2}\mathrm{trace}\, H(b)^2\right),$$

and $K = M(a_+^{-1})T(a_+) - I$.

Other results

An application of the above asymptotics yields an expansion for determinants of finite sections of operators of the form

$$T(a) \pm H(at^{\varkappa}).$$

By using the basic identity

$$\det PAP = (\det A) \cdot (\det QA^{-1}Q),$$

where $Q = I - P$ we can reduce these determinants to the previous cases and compute them asymptotically.

$\varkappa$ is a negative even integer ($\varkappa = -2\ell$, $\ell \geq 1$)

Suppose that $a = a_- a_0$, where a_0 is even and $a_- \in \overline{H^2}$. Then

$$\det P_N(T(a) \pm H(at^{\varkappa}))P_N$$

$$\sim G[a]^{N+\ell} E_{1,\pm}[a] \det P_\ell(T(a_0^{-1}) \pm H(a_0^{-1}))P_\ell$$

as $N \to \infty$, where $E_{1,\pm}[a]$ is given by

$$\exp\Big(\pm \sum_{n=1}^{\infty} \log a_{2n+1} - \frac{1}{2} \sum_{n=1}^{\infty} n[\log a]_n^2 + \sum_{n=1}^{\infty} n[\log a]_{-n}[\log a]_n \Big).$$

$\varkappa = -1 - 2\ell,\ \ell \geq 1$

Then

$$\det P_N(T(a) - H(at^{\varkappa}))P_N$$

$$\sim G[a]^{N+\ell} E_2[a] \det P_\ell(T(a_0^{-1}) - H(a_0^{-1}t^{-1})P_\ell$$

as $N \to \infty$, where $E_2[a]$ is given by

$$\exp\Big(-\sum_{n=1}^{\infty} \log a_{2n} - \frac{1}{2}\sum_{n=1}^{\infty} n[\log a]_n^2 + \sum_{n=1}^{\infty} n[\log a]_{-n}[\log a]_n\Big).$$

$\varkappa = 1 - 2\ell,\ \ell \geq 1$

Then

$$\det P_N(T(a) + H(at^{\varkappa}))P_N$$

$$\sim G[a]^{N+\ell} E_3[a] \det P_\ell(T(a_0^{-1}) + H(a_0^{-1}t))P_\ell$$

as $N \to \infty$, where $E_3[a]$ is given by

$$\exp\left(-\log 2 + \sum_{n=1}^{\infty} \log a_{2n} - \frac{1}{2}\sum_{n=1}^{\infty} n[\log a]_n^2 + \sum_{n=1}^{\infty} n[\log a]_{-n}[\log a]_n\right).$$

$\varkappa$ positive

We have

$$\begin{aligned} \det P_N(T(a) + H(at^{\varkappa}))P_N &= 0 \qquad \text{if } N \geq \varkappa \geq 2, \\ \det P_N(T(a) - H(at^{\varkappa}))P_N &= 0 \qquad \text{if } N \geq \varkappa \geq 1. \end{aligned}$$

How general can M be?

Let us write

$$K(a) = M(a) - T(a).$$

The main properties for compatible pairs implies the following:

Since $M(ab) = M(a)M(b)$ for b even, then

$$K(ab) = K(a)K(b) + T(a)K(b) + K(a)T(b) - H(a)H(b)$$

whenever b is even.

Also, $T(a)M(b) = M(ab)$ for $a \in \mathcal{S}_-$, implies that for $a \in \mathcal{S}_-$ we have

$K(a) = 0$ and $T(a)K(b) = K(ab)$ for any b.

Using these algebraic facts one can show that the structure of

M is determined by $K(t)$.

in fact

$$K(t) = e_0 x^T$$

where $e_0 x^T$ with $x \in \ell^2$ stand for the rank one operator

$$y \in \ell^2 \mapsto e_0 \langle y, x \rangle \in \ell^2.$$

and $e_0 = (1, 0, 0, \dots)$. The question of which x then generate an operator with the proper conditions is still not completely solved.