Truncated Moment Problems with Associated Finite Algebraic Varieties
(joint work with Seonguk Yoo)

Raúl Curto

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Dedicated to Bill on the occasion of his 65th birthday!
OUTLINE OF THE TALK

- Brief Review of Full Moment Problem
- Truncated Moment Problems (Basic Positivity, Functional Calculus, Algebraic Variety)
- Moment Matrix Extension Approach
- Positive Linear Functional Approach
- TMP Version of the Riesz-Haviland Theorem
- Structure of Positive Polynomials
- Cubic Column Relations
General Idea to Study TMP

- TMP is more general than FMP:
  fewer moments $\implies$ less data

- Stochel: link between TMP and FMP

- Existing approaches are directed at enlarging the data by acquiring new moments, and eventually making the problem into one of flat data type (i.e., with intrinsic recursiveness).

- This naturally leads to a full MP.

- If such a flat extension of the initial data cannot be accomplished, then TMP has no representing measure.

- Helpful tool: Smul’jan’s Theorem on positivity of $2 \times 2$ matrices
The Classical (Full) Moment Problem

Let $\beta \equiv \beta(\infty) = \{\beta_i\}_{i \in \mathbb{Z}^d_+}$ denote a $d$-dimensional real multisequence, and let $K$ (closed) $\subseteq \mathbb{R}^d$. The (full) $K$-moment problem asks for necessary and sufficient conditions on $\beta$ to guarantee the existence of a positive Borel measure $\mu$ supported in $K$ such that

$$\beta_i = \int x^i \, d\mu \quad (i \in \mathbb{Z}^d_+);$$

$\mu$ is called a rep. meas. for $\beta$.

Associated with $\beta$ is a moment matrix $M \equiv M(\infty)$, defined by

$$M_{ij} := \beta_{i+j} \quad (i, j \in \mathbb{Z}^d_+).$$
Basic Positivity Condition

$P_n$: polynomials $p$ over $\mathbb{R}$ with $\deg p \leq n$

Given $p \in P_n$, $p(x) \equiv \sum_{0 \leq i+j \leq n} a_i x^i$,

$$0 \leq \int p(x)^2 d\mu(x)$$

$$= \sum_{ij} a_i a_j \int x^{i+j} d\mu(x) = \sum_{ij} a_i a_j \beta_{i+j}.$$

Now recall that we’re working in $d$ real variables. To understand this “matricial” positivity, we introduce the following lexicographic order on the rows and columns of $M(n)$:

$$1, X_1, \ldots, X_d, X_1^2, X_2 X_1, \ldots, X_d^2, \ldots$$
Also recall that

\[ M(n)_{i,j} := \beta_{i+j}. \]

Then

\[
\sum_{ij} a_i a_j \beta_{i+j} \geq 0
\]

(“matricial” positivity) \[ \Leftrightarrow M(n) \equiv M(n)(\beta) \geq 0. \]
For example, for moment problems in $\mathbb{R}^2$,\[
M(1) = \begin{pmatrix}
\beta_{00} & \beta_{01} & \beta_{10} \\
\beta_{01} & \beta_{02} & \beta_{11} \\
\beta_{10} & \beta_{11} & \beta_{20}
\end{pmatrix},
\]
\[
M(2) = \begin{pmatrix}
\beta_{00} & \beta_{01} & \beta_{10} & \beta_{02} & \beta_{11} & \beta_{20} \\
\beta_{01} & \beta_{02} & \beta_{11} & \beta_{03} & \beta_{12} & \beta_{21} \\
\beta_{10} & \beta_{11} & \beta_{20} & \beta_{12} & \beta_{21} & \beta_{30} \\
\beta_{02} & \beta_{03} & \beta_{12} & \beta_{04} & \beta_{13} & \beta_{22} \\
\beta_{11} & \beta_{12} & \beta_{21} & \beta_{13} & \beta_{22} & \beta_{31} \\
\beta_{20} & \beta_{21} & \beta_{30} & \beta_{22} & \beta_{31} & \beta_{40}
\end{pmatrix}.
\]
In general,

\[ M(n + 1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix} \]

Similarly, one can build \( M(\infty) \equiv M(\infty)(\beta) \equiv M(\beta) \).

The link between TMP and FMP is provided by a result of Stochel (2001):

**Theorem (Stochel’s Theorem)**

\( \beta(\infty) \) has a rep. meas. supported in a closed set \( K \subseteq \mathbb{R}^d \) if and only if, for each \( n \), \( \beta^{(2n)} \) has a rep. meas. supported in \( K \).
\[ M := \{ \beta \equiv \beta(\infty) : \beta \text{ admits a rep. meas. } \mu \} \]
\[ B_+ := \{ \beta \equiv \beta(\infty) : M(\infty)(\beta) \geq 0 \} \]

Clearly, \( M \subseteq B_+ \)

- (Berg, Christensen and Ressel) \( \beta \in B_+ \), \( \beta \) bounded \( \Rightarrow \beta \in M \)
- (Berg and Maserick) \( \beta \in B_+ \), \( \beta \) exponentially bounded \( \Rightarrow \beta \in M \)
- (RC and L. Fialkow) \( \beta \in B_+ \), \( M(\beta) \) finite rank \( \Rightarrow \beta \in M \)
\( \mathcal{P}_+ : \) nonnegative poly’s
\( \Sigma^2 : \) sums of squares of poly’s

Clearly, \( \Sigma^2 \subseteq \mathcal{P}_+ \)

**Duality**

For \( C \) a cone in \( \mathbb{R}^{\mathbb{Z}^d}_+ \), we let

\[
C^*: = \{ \xi \in \mathbb{R}^{\mathbb{Z}^d}_+ : \text{supp}(\xi) \text{ is finite and } \langle p, \xi \rangle \geq 0 \text{ for all } p \in C \}.
\]

(Riesz-Haviland) \( \mathcal{P}_+^* = \mathcal{M} \)

For, consider the Riesz functional \( \Lambda_\beta(p) := p(\beta) \equiv \langle p, \beta \rangle \), which induces a map \( \mathcal{M} \rightarrow \mathcal{P}_+^* \) \( (\beta \mapsto \Lambda_\beta) \); Haviland’s Theorem says that this maps is onto, that is,

\[
\exists \mu \text{ rep. meas. for } \beta \iff \Lambda_\beta \geq 0 \text{ on } \mathcal{P}_+.
\]
\[ \mathcal{P}_+ = \mathcal{M}^* \text{ (straightforward once we have a r.m.)} \]

\[ \mathcal{B}_+ = (\Sigma^2)^* \text{ (straightforward)} \]

\[(\text{Berg, Christensen and Jensen}) \quad (\mathcal{B}_+)^* = \Sigma^2 \]

\[(n = 1) \quad \mathcal{P}_+ = \Sigma^2 \Rightarrow \mathcal{P}_+^* = (\Sigma^2)^* \Rightarrow \mathcal{M} = \mathcal{B}_+ \text{ (Hamburger)} \]

Generally, SOS implies the existence of a representing measure.
Consider the **full, complex** MP

\[ \int \bar{z}^i z^j \, d\mu = \gamma_{ij} \ (i, j \geq 0), \]

where \( \text{supp} \ \mu \subseteq K \), for \( K \) a closed subset of \( \mathbb{C} \).

- The **Riesz functional** is given by

  \[ \Lambda_{\gamma}(\bar{z}^i z^j) := \gamma_{ij} \ (i, j \geq 0). \]

- **Riesz-Haviland:**

  There exists \( \mu \) with \( \text{supp} \ \mu \subseteq K \iff \Lambda_{\gamma}(p) \geq 0 \) for all \( p \) such that \( p|_K \geq 0 \).
If \( q \) is a polynomial in \( z \) and \( \bar{z} \), and

\[
K \equiv K_q := \{ z \in \mathbb{C} : q(z, \bar{z}) \geq 0 \},
\]

then \( L_q(p) := L(qp) \) must satisfy \( L_q(p\bar{p}) \geq 0 \) for \( \mu \) to exist. For,

\[
L_q(p\bar{p}) = \int_{K_q} qp\bar{p} \, d\mu \geq 0 \quad (\text{all } p).
\]

- K. Schmüdgen (1991): If \( K_q \) is compact, \( \Lambda_\gamma(p\bar{p}) \geq 0 \) and \( L_q(p\bar{p}) \geq 0 \) for all \( p \), then there exists \( \mu \) with \( \text{supp } \mu \subseteq K_q \).
Given \( \gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{0,2n}, \ldots, \gamma_{2n,0}, \) with \( \gamma_{00} > 0 \) and \( \gamma_{ji} = \overline{\gamma_{ij}} \), the **TCMP** entails finding a positive Borel measure \( \mu \) supported in the complex plane \( \mathbb{C} \) such that

\[
\gamma_{ij} = \int \overline{z}^i z^j \, d\mu \quad (0 \leq i + j \leq 2n);
\]

\( \mu \) is called a **rep. meas.** for \( \gamma \).

In earlier joint work with L. Fialkow,

We have introduced an approach based on matrix positivity and extension, combined with a new “functional calculus” for the columns of the associated **moment matrix**.
We have shown that when the TCMP is of **flat data type**, a solution always exists; this is compatible with our previous results for

\[
\begin{align*}
\text{supp } \mu &\subseteq \mathbb{R} \quad \text{(Hamburger TMP)} \\
\text{supp } \mu &\subseteq [0, \infty) \quad \text{(Stieltjes TMP)} \\
\text{supp } \mu &\subseteq [a, b] \quad \text{(Hausdorff TMP)} \\
\text{supp } \mu &\subseteq \mathbb{T} \quad \text{(Toeplitz TMP)}
\end{align*}
\]

Along the way we have developed new machinery for analyzing TMP’s in **one or several real or complex variables**. For simplicity, in this talk we focus on **one complex variable or two real variables**, although several results have multivariable versions.
Our techniques also give concrete algorithms to provide finitely-atomic rep. meas. whose atoms and densities can be explicitly computed.

We have fully resolved, among others, the cases

$$\tilde{Z} = \alpha 1 + \beta Z$$

and

$$Z^k = p_{k-1}(Z, \tilde{Z}) \quad (1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil + 1; \deg p_{k-1} \leq k - 1).$$

We obtain applications to quadrature problems in numerical analysis.

We have obtained a duality proof of a generalized form of the Tchakaloff-Putinar Theorem on the existence of quadrature rules for positive Borel measures on $$\mathbb{R}^d$$. 
Applications

- Subnormal Operator Theory (unilateral weighted shifts)
- Physics (determination of contours)
- Computer Science (image recognition and reconstruction)
- Geography (location of proposed distribution centers)
- Probability (reconstruction of p.d.f.’s)
- Environmental Science (oil spills, via quadrature domains)
- Engineering (tomography)

- Geophysics (inverse problems, cross sections)

**Typical Problem:** Given a 3-D body, let X-rays act on the body at different angles, collecting the information on a screen. One then seeks to obtain a constructive, optimal way to approximate the body, or in some cases to reconstruct the body.
Positivity of Block Matrices

Theorem

\((\text{Smul’jan, 1959})\)

\[
\begin{pmatrix}
A & B \\
B^* & C
\end{pmatrix} \succeq 0 \iff \begin{cases}
A \succeq 0 \\
B = AW \\
C \succeq W^*AW
\end{cases}
\]

Moreover, \(\text{rank } \begin{pmatrix}
A & B \\
B^* & C
\end{pmatrix} = \text{rank } A \iff C = W^*AW.\)
**Corollary**

Assume $\text{rank } \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \text{rank } A$. Then

$$A \geq 0 \iff \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \succeq 0.$$
Basic Positivity Condition

\( \mathcal{P}_n \): polynomials \( p \) in \( z \) and \( \bar{z} \), \( \deg p \leq n \)

Given \( p \in \mathcal{P}_n \),

\[
p(z, \bar{z}) \equiv \sum_{0 \leq i+j \leq n} a_{ij} \bar{z}^i z^j,
\]

\[
0 \leq \int |p(z, \bar{z})|^2 \, d\mu(z, \bar{z})
= \sum_{i,j,k,l} a_{ij} \bar{a}_{kl} \int \bar{z}^{i+l} z^{j+k} \, d\mu(z, \bar{z})
= \sum_{i,j,k,l} a_{ij} \bar{a}_{kl} \gamma_{i+l, j+k}.
\]

To understand this “matricial” positivity, we introduce the following lexicographic order on the rows and columns of \( M(n) \):

\[
1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \ldots
\]
Define $M[i, j]$ as in

$$M[3, 2] := \begin{pmatrix}
\gamma_{32} & \gamma_{41} & \gamma_{50} \\
\gamma_{23} & \gamma_{32} & \gamma_{41} \\
\gamma_{14} & \gamma_{23} & \gamma_{32} \\
\gamma_{05} & \gamma_{14} & \gamma_{23}
\end{pmatrix}$$

Then

(“matricial” positivity) $\sum_{ijkl} a_{ij} \bar{a}_{k\ell} \gamma_{i+l,j+k} \geq 0$

$\Leftrightarrow M(n) \equiv M(n)(\gamma) := \begin{pmatrix}
M[0, 0] & M[0, 1] & \ldots & M[0, n] \\
M[1, 0] & M[1, 1] & \ldots & M[1, n] \\
\ldots & \ldots & \ldots & \ldots \\
M[n, 0] & M[n, 1] & \ldots & M[n, n]
\end{pmatrix} \geq 0.$
For example,

\[ M(1) = \begin{pmatrix}
\gamma_{00} & \gamma_{01} & \gamma_{10} \\
\gamma_{10} & \gamma_{11} & \gamma_{20} \\
\gamma_{01} & \gamma_{02} & \gamma_{11}
\end{pmatrix}, \]

\[ M(2) = \begin{pmatrix}
\gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\
\gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\
\gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\
\gamma_{20} & \gamma_{21} & \gamma_{12} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\
\gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\
\gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22}
\end{pmatrix}. \]
In general,

\[ M(n + 1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix} \]

Similarly, one can build \( M(\infty) \).

In the real case, \( M(n)_{ij} := \gamma_{i+j} \), \( i, j \in \mathbb{Z}_+^2 \).

**Positivity Condition is not sufficient:**

By modifying an example of K. Schmüdgen, we have built a family \( \gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{06}, \ldots, \gamma_{60} \) with positive invertible moment matrix \( M(3) \) but no rep. meas. But this can also be done for \( n = 2 \).
For $p \in P_n$, $p(z, \bar{z}) \equiv \sum_{0 \leq i+j \leq n} a_{ij} z^i \bar{z}^j$ define

$$p(Z, \bar{Z}) := \sum a_{ij} \bar{Z}^i Z^j \equiv M(n) \hat{p},$$

where $\hat{p} := (a_{00} \cdots a_{0n} \cdots a_{n0})^T$.

If there exists a rep. meas. $\mu$, then

$$p(Z, \bar{Z}) = 0 \iff \text{supp } \mu \subseteq \mathcal{Z}(p).$$

The following is our analogue of recursiveness for the TCMP

(RG) If $p, q, pq \in P_n$, and $p(Z, \bar{Z}) = 0$,

then $(pq)(Z, \bar{Z}) = 0.$
Given a finite family of moments, build moment matrix

Identify all column relations

Build algebraic variety $\mathcal{V}$

Always true:

$$r := \text{rank } M(n) \leq \text{card } \text{supp } \mu \leq \nu := \text{card } \mathcal{V}(\gamma),$$

so if the variety is finite there’s a natural candidate for $\text{supp } \mu$, i.e.,

$\text{supp } \mu = \mathcal{V}(\gamma)$
Finite rank case
Flat case
Extremal case
Recursively generated relations

Strategy: Build positive extension, repeat, and eventually extremal

\[ \text{rank } M(n) \leq \text{rank } M(n + 1) \leq \text{card } \mathcal{V}(M(n + 1)) \leq \text{card } \mathcal{V}(M(n)) \]

General case.
**First Existence Criterion**

**Theorem**

*(RC-L. Fialkow, 1998)* Let $\gamma$ be a truncated moment sequence. TFAE:

(i) $\gamma$ has a rep. meas.;

(ii) $\gamma$ has a rep. meas. with moments of all orders;

(iii) $\gamma$ has a compactly supported rep. meas.;

(iv) $\gamma$ has a finitely atomic rep. meas. (with at most $(n+2)(2n+3)$ atoms);

(v) $M(n) \geq 0$ and for some $k \geq 0$ $M(n)$ admits a positive extension $M(n+k)$, which in turn admits a flat (i.e., rank-preserving) extension $M(n+k+1)$ (here $k \leq 2n^2 + 6n + 6$).
Case of Flat Data

Recall: If $\mu$ is a rep. meas. for $M(n)$, then rank $M(n) \leq \text{card supp} \mu$.

$\gamma$ is flat if $M(n) = \begin{pmatrix} M(n-1) & M(n-1)W \\ W^*M(n-1) & W^*M(n-1)W \end{pmatrix}$.

Theorem (RC-L. Fialkow, 1996) If $\gamma$ is flat and $M(n) \geq 0$, then $M(n)$ admits a unique flat extension of the form $M(n+1)$.

Theorem (RC-L. Fialkow, 1996) The truncated moment sequence $\gamma$ has a rank $M(n)$-atomic rep. meas. if and only if $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n+1)$.

To find $\mu$ concretely, let $r := \text{rank } M(n)$ and look for the relation
\[ Z^r = c_0 1 + c_1 Z + \ldots + c_{r-1} Z^{r-1}. \]

We then define
\[ p(z) := z^r - (c_0 + \ldots + c_{r-1} z^{r-1}) \]
and solve the Vandermonde equation
\[
\begin{pmatrix}
1 & \ldots & 1 \\
z_0 & \ldots & z_{r-1} \\
\vdots & \ddots & \vdots \\
z_0^{r-1} & \ldots & z_{r-1}^{r-1}
\end{pmatrix}
\begin{pmatrix}
\rho_0 \\
\rho_1 \\
\vdots \\
\rho_{r-1}
\end{pmatrix}
= 
\begin{pmatrix}
\gamma_{00} \\
\gamma_{01} \\
\vdots \\
\gamma_{0r-1}
\end{pmatrix}.
\]

Then
\[ \mu = \sum_{j=0}^{r-1} \rho_j \delta_{z_j}. \]
Recall the lexicographic order on the rows and columns of $M(2)$:

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2$$

- $Z = A \cdot 1$ (Dirac measure)
- $\bar{Z} = A \cdot 1 + B \cdot Z$ (supp $\mu \subseteq$ line)
- $Z^2 = A \cdot 1 + B \cdot Z + C \cdot \bar{Z}$ (flat extensions always exist)
- $\bar{Z}Z = A \cdot 1 + B \cdot Z + C \cdot \bar{Z} + D \cdot Z^2$

$$D = 0 \Rightarrow \bar{Z}Z = A \cdot 1 + B \cdot Z + \bar{B} \cdot \bar{Z} \text{ and } C = \bar{B}$$

$$\Rightarrow (\bar{Z} - B)(Z - \bar{B}) = A + |B|^2$$

$$\Rightarrow \bar{W}W = 1 \text{ (circle), for } W := \frac{Z - \bar{B}}{\sqrt{A + |B|^2}}.$$
The functional calculus we have constructed is such that \( p(Z, \bar{Z}) = 0 \) implies \( \text{supp} \ \mu \subseteq \mathcal{Z}(p) \).

When \( \{1, Z, \bar{Z}, Z^2, \bar{Z}Z\} \) is a basis for \( \mathcal{C}_{M(2)} \), the associated algebraic variety is the zero set of a real quadratic equation in

\[
x := \text{Re}[z] \text{ and } y := \text{Im}[z].
\]

Using the flat data result, one can reduce the study to cases corresponding to the following four real conics:

\[
\begin{align*}
(a) \quad \bar{W}^2 &= -2iW + 2i\bar{W} - W^2 - 2\bar{W}W \quad \text{parabola; } y = x^2 \\
(b) \quad \bar{W}^2 &= -4i1 + W^2 \quad \text{hyperbola; } yx = 1 \\
(c) \quad \bar{W}^2 &= W^2 \quad \text{pair of intersect. lines; } yx = 0 \\
(d) \quad \bar{W}W &= 1 \quad \text{unit circle; } x^2 + y^2 = 1.
\end{align*}
\]
**Theorem QUARTIC**

(RC-L. Fialkow, 2005) Let $\gamma^{(4)}$ be given, and assume $M(2) \geq 0$ and 
$\{1, Z, \bar{Z}, Z^2, \bar{Z}Z\}$ is a basis for $C_{\mathcal{M}(2)}$. Then $\gamma^{(4)}$ admits a rep. meas. $\mu$. Moreover, it is possible to find $\mu$ with $\text{card sup} \mu = \text{rank } M(2)$, except in some cases when $\mathcal{V}(\gamma^{(4)})$ is a pair of intersecting lines, in which cases there exist $\mu$ with $\text{card sup} \mu \leq 6$.

**Corollary**

Assume that $M(2) \geq 0$ and that $\text{rank } M(2) \leq \text{card } \mathcal{V}(\gamma^{(4)})$. Then $M(2)$ admits a representing measure.
The algebraic variety of $\gamma$ is

$$\mathcal{V} \equiv \mathcal{V}(\gamma) := \bigcap_{p \in \mathcal{P}_n, \hat{p} \in \ker M(n)} \mathcal{Z}_p,$$

where $\mathcal{Z}_p$ is the zero set of $p$.

If $\gamma$ admits a representing measure $\mu$, then

$$p \in \mathcal{P}_n \text{ satisfies } M(n)\hat{p} = 0 \iff \text{supp } \mu \subseteq \mathcal{Z}_p$$

Thus $\text{supp } \mu \subseteq \mathcal{V}$, so $r := \text{rank } M(n)$ and $v := \text{card } \mathcal{V}$ satisfy

$$r \leq \text{card supp } \mu \leq v.$$

If $p \in \mathcal{P}_{2n}$ and $p|_{\mathcal{V}} \equiv 0$, then $\Lambda(p) = \int p \, d\mu = 0$.

Here $\Lambda$ is the Riesz functional, given by $\Lambda(\bar{z}^i z^j) := \gamma_{ij}$.
Basic necessary conditions for the existence of a representing measure

(Positivity) \( M(n) \geq 0 \) \hspace{1cm} (9.1)

(Consistency) \( p \in \mathcal{P}_{2n}, \ p|_\mathcal{V} \equiv 0 \implies \Lambda(p) = 0 \) \hspace{1cm} (9.2)

(Variety Condition) \( r \leq \nu, \ \text{i.e., rank} \ M(n) \leq \text{card} \ \mathcal{V} \). \hspace{1cm} (9.3)

Consistency implies

(Recursiveness) \( p, q, pq \in \mathcal{P}_n, \ M(n)\hat{p} = 0 \implies M(n)(pq)\hat{=} = 0 \). \hspace{1cm} (9.4)
Previous results:

- For $d = 1$ (the T \textit{Hamburger} MP for $\mathbb{R}$), positivity and recursiveness are sufficient.
- For $d = 2$, there exists $M(3) > 0$ for which $\gamma$ has no representing measure.
- In general, \textit{Positivity, Consistency} and the \textit{Variety Condition} are not sufficient.

\textbf{Question C}

\textit{Suppose $M(n)(\gamma)$ is singular. If $M(n)$ is positive, $\gamma$ is consistent, and $r \leq v$, does $\gamma$ admit a representing measure?}
The next result gives an affirmative answer to Question C in the extremal case, i.e., $r = v$.

**Theorem EXT**

$(RC, L. Fialkow and M. Möller, 2005)$ For $\gamma \equiv \gamma^{(2n)}$ extremal, i.e., $r = v$, the following are equivalent:

(i) $\gamma$ has a representing measure;
(ii) $\gamma$ has a unique representing measure, which is rank $M(n)$-atomic (minimal);
(iii) $M(n) \geq 0$ and $\gamma$ is consistent.
Since we know how to solve the singular Quartic MP, WLOG we will assume $M(2) > 0$.

Recall

**Theorem A**

*(RC-L. Fialkow)* If $M(n)$ admits a column relation of the form

$$Z^k = p_{k-1}(Z, \bar{Z}) \quad (1 \leq k \leq \left[ \frac{n}{2} \right] + 1 \text{ and } \deg p_{k-1} \leq k - 1),$$

then $M(n)$ admits a flat extension $M(n+1)$, and therefore a representing measure.

Now, if $k = 3$, Theorem A can be used only if $n \geq 4$. Thus, one strategy is to somehow extend $M(3)$ to $M(4)$ and preserve the column relation $Z^3 = p_2(Z, \bar{Z})$. This requires checking that the $C$ block in the extension satisfies the Toeplitz condition, something highly nontrivial.
Here’s a different approach: We’d like to study the case of harmonic poly’s: \( q(z, \bar{z}) := f(z) - \bar{g(z)} \), with \( \text{deg } q = 3 \). Recall that \( \text{rank } M(n) \leq \text{card } \mathcal{Z}(q) \) so of special interest is the case when \( \text{card } \mathcal{Z}(q) \geq 7 \), since otherwise the TMP admits a flat extension, or has no representing measure. In the case when \( g(z) \equiv z \), we have

**Lemma**

\( (\text{Wilmshurst '98, Sarason-Crofoot, '99, Khavinson-Swiatek, '03}) \)

\[ \text{card } \mathcal{Z}(f(z) - \bar{z}) \leq 7. \]
To get 7 points is not easy, as most complex cubic harmonic poly’s tend to have 5 or fewer zeros. One way to maximize the number of zeros is to impose symmetry conditions on the zero set $K$. Also, the substitution $w = z + b/3$ (which produces an equivalent TMP) transforms a cubic $z^3 + bz^2 + cz + d$ into $w^3 + \tilde{c}w + \tilde{d}$; WLOG, we always assume that there’s no quadratic term in the analytic piece.

Now, for a poly of the form $z^3 + \alpha z + \beta \bar{z}$, it is clear that $0 \in K$ and that $z \in K \Rightarrow -z \in K$. Another natural condition is to require that $K$ be symmetric with respect to the line $y = x$, which in complex notation is $z = i\bar{z}$. When this is required, we obtain $\alpha \in i\mathbb{R}$ and $\beta \in \mathbb{R}$. Thus, the column relation becomes $Z^3 = itZ + u\bar{Z}$, with $t, u \in \mathbb{R}$.

Under these conditions, one needs to find only two points, one on the line $y = x$, the other outside that line.
We thus consider the harmonic polynomial $q_7(z, \bar{z}) := z^3 - itz - u\bar{z}$.

**Proposition**

(RC-S. Yoo, ’09) Card $\mathcal{Z}(q_7) = 7$. In fact, for $0 < |u| < t < 2|u|$, $\mathcal{Z}(q_7) = \{0, p + iq, q + ip, -p - iq, -q - ip, r + ir, -r - ir\}$, where $p, q, r > 0$, $p^2 + q^2 = u$ and $r^2 = \frac{t-u}{2}$. 
To prove this result, we first identify the two real poly’s

\[ \text{Re } q_7 = x^3 - 3xy^2 + ty - ux \] and
\[ \text{Im } q_7 = -y^3 + 3x^2y - tx + uy \]

and calculate \( \text{Resultant}(\text{Re}q_7, \text{Im}q_7, y) \), which is the determinant of the Sylvester matrix, i.e.,

\[
\begin{vmatrix}
-3x & t & x^3 - ux & 0 & 0 \\
0 & -3x & t & x^3 - ux & 0 \\
0 & 0 & -3x & t & x^3 - ux \\
-1 & 0 & 3x^2 + u & -tx & 0 \\
0 & -1 & 0 & 3x^2 + u & -tx \\
\end{vmatrix}
= x (u - t + 2x^2) (u + t + 2x^2) (16x^4 - 16x^2u + t^2).
\]
Figure 1. The 7-point set $Z(q_7)$, where

$$r = \sqrt{\frac{t-u}{2}}, \quad p = \frac{1}{2}(2u + \sqrt{4u^2 - t^2}) \quad \text{and} \quad p^2 + q^2 = u$$
The fact that \( q_7 \) has the \textbf{maximum} number of zeros predicted by the Lemma is significant to us, in that each \textbf{sextic} TMP with \textit{invertible} \( M(2) \) and a column relation of the form \( q_7(Z, \bar{Z}) = 0 \) either does not admit a representing measure or is necessarily extremal.

As a consequence, the existence of a representing measure will be established once we prove that such a TMP is \textit{consistent}. This means that for each poly \( p \) of degree at most 6 that vanishes on \( Z(q_7) \) we must verify that \( \Lambda(p) = 0 \).
Since rank $M(3) = 7$, there must be another column relation besides $q_7(Z, \bar{Z}) = 0$. Clearly the columns

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \bar{Z}Z^2$$

must be linearly independent (otherwise $M(3)$ would be a flat extension of $M(2)$), so the new column relation must involve $\bar{Z}Z^2$ and $\bar{Z}^2Z$. An analysis using the properties of the functional calculus shows that, in the presence of a representing measure, the new column relation must be

$$\bar{Z}^2Z + i\bar{Z}Z^2 - iuZ - u\bar{Z} = 0.$$
Notation

In what follows, \( \mathbb{C}_6[z, \bar{z}] \) will denote the space of complex polynomials in \( z \) and \( \bar{z} \) of degree at most 6, and let

\[
q_{LC}(z, \bar{z}) := \bar{z}^2z + i\bar{z}z^2 - iuz - u\bar{z} = i(z - i\bar{z})(\bar{z}z - u).
\]

Observe that the zero set of \( q_{LC} \) is the union of a line and a circle, and that \( \mathcal{Z}(q_7) \subset \mathcal{Z}(q_{LC}) \).
Figure 2. The sets $\mathcal{Z}(q_7)$ and $\mathcal{Z}(q_{LC})$. 
Main Theorem

Let $M(3) \geq 0$, with $M(2) > 0$ and $q_7(Z, \bar{Z}) = 0$. There exists a representing measure for $M(3)$ if and only if

\[
\begin{align*}
\Lambda(q_{LC}) &= 0 \\
\Lambda(zq_{LC}) &= 0.
\end{align*}
\]  

(11.1)

Equivalently,

\[
\begin{align*}
\text{Re} \gamma_{12} - \text{Im} \gamma_{12} &= u(\text{Re} \gamma_{01} - \text{Im} \gamma_{01}) = 0 \\
\gamma_{22} &= (t + u)\gamma_{11} - 2u \text{ Im} \gamma_{02} = 0.
\end{align*}
\]

Equivalently,

$q_{LC}(Z, \bar{Z}) = 0$  

(11.2)
Proof. $(\Longleftrightarrow)$ Let $\mu$ be a representing measure. We know that 
$7 \leq \text{rank } M(3) \leq \text{card } \text{supp } \mu \leq \text{card } \mathcal{Z}(q_7) = 7$, so that 
$\text{supp } \mu = \mathcal{Z}(q_7)$ and $\text{rank } M(3) = 7$. Thus,

$$\Lambda(q_7) = \int q_7 \, d\mu = 0.$$

Similarly, since $\text{supp } \mu \subseteq \mathcal{Z}(q_{LC})$, we also have

$$\Lambda(q_{LC}) = \Lambda(zq_{LC}) = 0,$$

as desired.
(⇐) On \( \mathcal{Z}(q_7) \) we have \( z^3 = itz + u\bar{z} \). Using this relation and (11.1), we can prove that \( \Lambda(\bar{z}^i z^j q_{LC}) = 0 \) for all \( 0 \leq i + j \leq 3 \). For example,

\[
\bar{z}q_{LC} - izq_{LC} = (\bar{z} - iz)(\bar{z}^2z + i\bar{z}z^2 - iuz - u\bar{z})
\]
\[
= -uz^2 + \bar{z}z^3 - u\bar{z}^2 + \bar{z}^3z
\]
\[
= -uz^2 + \bar{z}(itz + u\bar{z}) - u\bar{z}^2 + (-itz + uz)z
\]
\[
= 0,
\]
and therefore \( \Lambda(\bar{z}q_{LC}) = i\Lambda(zq_{LC}) = 0 \). It follows that for \( f, g, h \in \mathbb{C}_3[z, \bar{z}] \) we have \( \Lambda(fq_7 + g\bar{q}_7 + hq_{LC}) = 0 \). Consistency will be established once we show that all degree-six polynomials vanishing in \( \mathcal{Z}(q_7) \) are of the form \( fq_7 + g\bar{q}_7 + hq_{LC} \).
Proposition (Representation of Polynomials)

Let $\mathcal{P}_6 := \{ p \in \mathbb{C}_6[z, \bar{z}] : p|_{z(q_7)} \equiv 0 \}$ and let

$\mathcal{I} := \{ p \in \mathbb{C}_6[z, \bar{z}] : p = fq_7 + g\bar{q}_7 + hq_{LC} \text{ for some } f, g, h \in \mathbb{C}_3[z, \bar{z}] \}.$

Then $\mathcal{P}_6 = \mathcal{I}$.

Proof. Clearly, $\mathcal{I} \subseteq \mathcal{P}_6$. We shall show that $\dim \mathcal{I} = \dim \mathcal{P}_6$. Let $T : \mathbb{C}^{30} \longrightarrow \mathbb{C}_6[z, \bar{z}]$ be given by

$$(a_{00}, \ldots, a_{30}, b_{00}, \ldots, b_{30}, c_{00}, \ldots, c_{30}) \longmapsto$$

$$(a_{00} + a_{01}z + a_{10}\bar{z} + \cdots + a_{30}\bar{z}^3)q_7$$

$$+(b_{00} + b_{01}z + b_{10}\bar{z} + \cdots + b_{30}\bar{z}^3)\bar{q}_7$$

$$+(c_{00} + c_{01}z + c_{10}\bar{z} + \cdots + c_{30}\bar{z}^3)q_{LC}.$$
Recall that $30 = \dim \mathbb{C}^{30} = \dim \ker T + \dim \Ran T$, and observe that $\mathcal{I} = \Ran T$, so that $\dim \mathcal{I} = \rank T$.

To determine $\rank T$, we first determine $\dim \ker T$. Using Gaussian elimination, we prove that $\dim \ker T = 9$ whenever $ut \neq 0$. It follows that $\rank T = 30 - 9 = 21$, that is, $\dim \mathcal{I} = 21$. 
Now consider the evaluation map $S : \mathbb{C}_6[z, \bar{z}] \longrightarrow \mathbb{C}^7$ given by

$$S(p(z, \bar{z})) := (p(w_0, \bar{w}_0), p(w_1, \bar{w}_1), p(w_2, \bar{w}_2),$$

$$p(w_3, \bar{w}_3), p(w_4, \bar{w}_4), p(w_5, \bar{w}_5), p(w_6, \bar{w}_6)).$$

Again, $\dim \ker S + \dim \text{Ran } S = \dim \mathbb{C}_6[z, \bar{z}] = 28$. Using Lagrange Interpolation, it is easy to verify that $S$ is onto, i.e., $\text{rank } S = 7$.

Moreover, $\ker S = \mathcal{P}_6$. Since $\dim \mathbb{C}_6[z, \bar{z}] = 28$, it follows that $\dim \ker S = 21$, and a fortiori that $\dim \mathcal{P}_6 = 21$.

Therefore, $\dim \mathcal{I} = 21 = \dim \mathcal{P}_6$, and since $\mathcal{I} \subseteq \mathcal{P}_6$, we have established that $\mathcal{I} = \mathcal{P}_6$, as desired.
Yet another approach to TMP: The Division Algorithm

**Division Algorithm in** $\mathbb{R}[x_1, \cdots, x_n]$

Fix a monomial order $>$ on $\mathbb{Z}^n_{\geq 0}$ and let $F = (f_1, \cdots, f_s)$ be an ordered $s$-tuple of polynomials in $\mathbb{R}[x_1, \cdots, x_n]$. Then every $f \in \mathbb{R}[x_1, \cdots, x_n]$ can be written as

$$f = a_1 f_1 + \cdots + a_s f_s + r,$$

where $a_i \in \mathbb{R}[x_1, \cdots, x_n]$, and either $r = 0$ or $r$ is a linear combination, with coefficients in $\mathbb{R}$, of monomials, none of which is divisible by any of the leading terms in $f_1, \cdots, f_s$.

Furthermore, if $a_i f_i \neq 0$, then we have

$$\text{multideg}(f) \geq \text{multideg}(a_i f_i).$$
With S. Yoo, we have recently used the Division Algorithm to build an example of $M(3) \equiv M(3)(\beta) \geq 0$ satisfying $M(2) > 0$, $r = 7$, $\nu = \infty$, $\beta$ consistent, and with no representing measure.

The Division Algorithm work is as follows: we identify sufficiently many polynomials $f_1, \cdots, f_s$ vanishing on $V(\beta)$, and simultaneously in the kernel of the Riesz functional $L_\beta$. By the Division Algorithm, any polynomial $f$ vanishing on $V(\beta)$ can be written as $f = a_1f_1 + \cdots + a_s f_s + r$, which readily implies that $r$ must also vanish on $V(\beta)$. Due to the divisibility condition on the monomials of $r$, and the characteristics of $V(\beta)$, which generate an invertible Vandermonde matrix, we then prove that $r \equiv 0$.

With some additional work, it is then possible to prove that $f \in \ker L_\beta$, which establishes the Consistency of $\beta$. 
Given a finite family of moments, build moment matrix

Identify all column relations, and build algebraic variety $\mathcal{V}$

Always true: $r \leq \text{card supp } \mu \leq \nu$

Finite rank case; flat case

Quartic Case

Extremal case (must check Consistency)

Harmonic cubic poly’s in Sextic Case

General singular case

Invertible case still a big mystery...