Noncommutative functions: Algebraic and analytic results

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October 3, 2010
Bill Helton Workshop, UCSD

1 A joint work with V. Vinnikov
For a vector space $\mathcal{V}$ over a field $\mathbb{K}$, we define the *noncommutative* (nc) space over $\mathcal{V}$

$$\mathcal{V}_{nc} = \bigoplus_{n=1}^{\infty} \mathcal{V}^{n \times n}.$$  

For $X \in \mathcal{V}^{n \times n}$ and $Y \in \mathcal{V}^{m \times m}$ we define their direct sum

$$X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \mathcal{V}^{(n+m) \times (n+m)}.$$

Notice that matrices over $\mathbb{K}$ act from the right and from the left on matrices over $\mathcal{V}$ by the standard rules of matrix multiplication: if $X \in \mathcal{V}^{p \times q}$ and $T \in \mathbb{K}^{r \times p}$, $S \in \mathbb{K}^{q \times s}$, then

$$TX \in \mathcal{V}^{r \times q}, \quad XS \in \mathcal{V}^{p \times s}.$$  

A subset $\Omega \subseteq \mathcal{V}_{nc}$ is called a *nc set* if it is closed under direct sums; explicitly, denoting $\Omega_n = \Omega \cap \mathcal{V}^{n \times n}$, we have $X \oplus Y \in \Omega_{n+m}$ for all $X \in \Omega_n$, $Y \in \Omega_m$. 
In the case of $\mathcal{V} = \mathbb{K}^d$ we identify matrices over $\mathcal{V}$ with $d$-tuples of matrices over $\mathbb{K}$:

$$
\left( \mathbb{K}^d \right)^{p \times q} \cong \left( \mathbb{K}^{p \times q} \right)^d.
$$

Under this identification, for $d$-tuples $X = (X_1, \ldots, X_d) \in \left( \mathbb{K}^{n \times n} \right)^d$ and $Y = (Y_1, \ldots, Y_d) \in \left( \mathbb{K}^{m \times m} \right)^d$,

$$
X \oplus Y = \left( \begin{bmatrix} X_1 & 0 \\ 0 & Y_1 \end{bmatrix}, \ldots, \begin{bmatrix} X_d & 0 \\ 0 & Y_d \end{bmatrix} \right) \in \left( \mathbb{K}^{(n+m) \times (n+m)} \right)^d,
$$

and for a $d$-tuple $X = (X_1, \ldots, X_d) \in \left( \mathbb{K}^{p \times q} \right)^d$ and matrices $T \in \mathbb{K}^{r \times p}$, $S \in \mathbb{K}^{q \times s}$,

$$
TX = (TX_1, \ldots, TX_d) \in \left( \mathbb{K}^{r \times q} \right)^d, \quad XS = (X_1S, \ldots, X_dS) \in \left( \mathbb{K}^{p \times s} \right)^d.
$$
Let \( \mathcal{V} \) and \( \mathcal{W} \) be vector spaces over \( \mathbb{K} \), and let \( \Omega \subseteq \mathcal{V}_{\text{nc}} \) be a nc set. A mapping \( f : \Omega \to \mathcal{W}_{\text{nc}} \) with \( f(\Omega_n) \subseteq \mathcal{W}^{n \times n} \) is called a \textit{nc function} if \( f \) satisfies the following two conditions:

- **f respects direct sums:**
  \[
  f(X \oplus Y) = f(X) \oplus f(Y), \quad X, Y \in \Omega. \tag{1}
  \]

- **f respects similarities:** if \( X \in \Omega_n \) and \( S \in \mathbb{K}^{n \times n} \) is invertible with \( SXS^{-1} \in \Omega_n \), then
  \[
  f(SXS^{-1}) = Sf(X)S^{-1}. \tag{2}
  \]

**Proposition**

A mapping \( f : \Omega \to \mathcal{W}_{\text{nc}} \) with \( f(\Omega_n) \subseteq \mathcal{W}^{n \times n} \) respects direct sums and similarities, i.e., (1) and (2) hold \textit{iff} \( f \) respects intertwinnings: for any \( X \in \Omega_n \), \( Y \in \Omega_m \), and \( T \in \mathbb{K}^{n \times m} \) such that \( XT = TY \),

\[
  f(X)T = Tf(Y). \tag{3}
  \]
(a) **NC polynomials and nc rational expressions.**
In many engineering applications, matrices appear as natural variables. Stability problems in control theory are usually reduced to Stein, Lyapunov, or Riccati equations or inequalities where the left-hand side is a nc polynomial, e.g., as in the continuous-time Riccati inequality

\[ p(X, A, A^*, B, C) := AX + XA^* + XBX + C \leq 0, \]

or a nc rational expression, as in the discrete-time Riccati inequality

+ (C^*D + A^*XB)(I - D^*D - B^*XB)^{-1}(D^*C + B^*XA) \leq 0. \]

Other polynomial and rational matrix inequalities arise in optimization and related problems, such as nc sum-of-squares (SoS) representations of positive nc polynomials, factorization of hereditary polynomials, nc positivestellensatz, matrix convexity, etc. [Helton, McCullough, Putinar, ...].
(b) **NC formal power series.**

Let $\mathcal{F}_d$ denote the free semigroup with $d$ generators $g_1, \ldots, g_d$ (the letters) and unity $\emptyset$ (the empty word). For a word $w = g_{j_1} \cdots g_{j_m} \in \mathcal{F}_d$ its length is $|w| = m \in \mathbb{N}$, and $|\emptyset| = 0$. Let $z_1, \ldots, z_d$ be noncommuting indeterminates and $w = g_{j_1} \cdots g_{j_m} \in \mathcal{F}_d$. Set $z^w = z_{j_1} \cdots z_{j_m}$, $z^\emptyset = 1$. For a linear space $\mathcal{L}$, the formal power series (FPSs) in $z_1, \ldots, z_d$ with coefficients in $\mathcal{L}$ has the form

$$f(z) = \sum_{w \in \mathcal{F}_d} f_w z^w.$$

NC polynomials are finite FPSs:

$$p(z) = \sum_{w \in \mathcal{F}_d : |w| \leq m} p_w z^w.$$
One can evaluate FPSs on $d$-tuples of bounded linear operators (or of $n \times n$ matrices, $n = 1, 2 \ldots$):

$$f(X) = \sum_{w \in F_d} f_w \otimes X^w$$

(of course, provided this series converges in certain topology).

NC FPSs appear in NC algebra, finite automata and formal languages, enumeration combinatorics, probability, system theory...

(c) Quasideterminants, nc symmetric functions [Gelfand–Retakh...].

(d) NC continued fractions [Wedderburn].

(e) Formal Baker–Campbell–Hausdorff series [Dynkin].

(f) Analytic functions of several noncommuting variables. [J. L. Taylor]

Some applications in free probability [Voiculescu].
Theorem

Let $f : \Omega \rightarrow \mathcal{W}_{nc}$ be a nc function on a nc set $\Omega$. Let $X \in \Omega_n$, $Y \in \Omega_m$, and $Z \in \mathcal{V}^{n \times m}$ be such that $\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}$. Then

$$f \left( \begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \right) = \begin{bmatrix} f(X) & \Delta_R f(X, Y)(Z) \\ 0 & f(Y) \end{bmatrix},$$

where the off-diagonal block entry $\Delta_R f(X, Y)(Z)$ is determined uniquely and is linear in $Z$.

$\Delta_R$ plays a role of a right difference-differential operator. Thus, the formula of finite differences holds:

$$f(X) - f(Y) = \Delta_R f(Y, X)(X - Y) \quad n \in \mathbb{N}, \; X, Y \in \Omega_n.$$  

The linear mapping $\Delta f(Y, Y)(\cdot)$ plays the role of a nc differential. If $K = \mathbb{R}$ or $\mathbb{C}$, setting $X = Y + tZ$ with $t \in \mathbb{R}$ ($t \in \mathbb{C}$), we obtain

$$f(Y + tZ) - f(Y) = t\Delta_R f(Y, Y + tZ)(Z).$$
Under appropriate continuity conditions, it follows that $\Delta_R f(Y, Y)(Z)$ is the directional derivative of $f$ at $Y$ in the direction $Z$.

In the case of $\mathcal{V} = \mathbb{K}^d$, the finite difference formula turns into

$$f(X) - f(Y) = \sum_{i=1}^{N} \Delta_{R,i} f(Y, X)(X_i - Y_i), \quad X, Y \in \Omega_n,$$

with the right partial difference-differential operators $\Delta_{R,i}$:

$$\Delta_{R,i} f(Y, X)(C) := \Delta_R f(Y, X)(0, \ldots, 0, C, 0, \ldots, 0).$$

The linear mapping $\Delta_{R,i} f(Y, Y)(\cdot)$ plays the role of a right nc $i$-th partial differential at the point $Y$.

The left nc full and partial difference-differential operators $\Delta_L$, $\Delta_{L,i}$, $i = 1, \ldots, d$, are defined analogously.
Properties of $\Delta_R$ (the calculus rules).

1. If $c \in \mathcal{W}$ and $f(X) = c \otimes I_n$, then $\Delta_R f(X, Y)(Z) = 0$.
2. $\Delta_R(af + bg) = a\Delta_R f + b\Delta_R g$ for any $a, b \in \mathbb{K}$.
3. If $\ell: \mathcal{V} \to \mathcal{W}$ is linear, then it can be extended to $\ell: \mathcal{V}^{n \times m} \to \mathcal{W}^{n \times m}$ by $\ell([v_{ij}]) = [\ell(v_{ij})]$. Then
   \[ \Delta_R \ell(X, Y)(Z) = \ell(Z). \]

   In particular, if $\ell_j: \mathbb{K}_{nc}^d \to \mathbb{K}_{nc}$ is the $j$-th coordinate nc function, i.e., $\ell_j(X) = \ell_j(X_1, \ldots, X_d) = X_j$, then
   \[ \Delta_R \ell_j(X, Y)(Z) = Z_j. \]

4. If $f: \Omega \to \mathcal{X}_{nc}$, $g: \Omega \to \mathcal{Y}_{nc}$ be nc functions. Assume that the product $(x, y) \mapsto x \cdot y$ on $\mathcal{X} \times \mathcal{Y}$ with values in a vector space $\mathcal{W}$ over $\mathbb{K}$ is well defined. We extend the product to matrices over $\mathcal{X}$ and over $\mathcal{Y}$ of appropriate sizes. Then
   \[ \Delta_R(f \cdot g)(X, Y)(Z) = f(X) \cdot \Delta_R g(X, Y)(Z) + \Delta_R f(X, Y)(Z) \cdot g(Y). \]
5. Let \( f : \Omega \to \mathcal{A}_{\text{nc}} \) be a nc function, where \( \mathcal{A} \) is a unital algebra over \( \mathbb{K} \). Let

\[
\Omega^{\text{inv}} := \bigsqcup_{n=1}^{\infty} \{ X \in \Omega_n : f(X) \text{ is invertible in } \mathcal{A}^{n \times n} \}.
\]

Then \( \Omega^{\text{inv}} \) is a nc set, \( f^{-1} : \Omega^{\text{inv}} \to \mathcal{A}_{\text{nc}} \) defined by \( f^{-1}(X) := f(X)^{-1} \) is a nc function, and

\[
\Delta_{R}f^{-1}(X, Y)(Z) = -f(X)^{-1}\Delta_{R}f(X, Y)(Z)f(Y)^{-1}.
\]

6. Let \( f : \Omega \to \mathcal{X}_{\text{nc}} \) and \( g : \Lambda \to \mathcal{Y}_{\text{nc}} \) be nc functions on nc sets \( \Omega \subseteq \mathcal{Y}_{\text{nc}} \) and \( \Lambda \subseteq \mathcal{X}_{\text{nc}} \), such that \( f(\Omega) \subseteq \Lambda \). Then the composition \( g \circ f : \Omega \to \mathcal{Y}_{\text{nc}} \) is a nc function, and

\[
\Delta_{R}(g \circ f)(X, Y)(Z) = \Delta_{R}g(f(X), f(Y))(\Delta_{R}f(X, Y)(Z)).
\]
As a function of $X$ and $Y$, $\Delta_R f(X, Y)(\cdot)$ respects direct sums and similarities, or equivalently, respects intertwinings: if $X \in \Omega_n$, $Y \in \Omega_m$, $\tilde{X} \in \Omega_{\tilde{n}}$, $\tilde{Y} \in \Omega_{\tilde{m}}$, and $T \in K^{\tilde{n} \times n}$, $S \in K^{m \times \tilde{m}}$ are such that

$$TX = \tilde{X}T, \quad YS = S\tilde{Y},$$

then

$$T\Delta_R f(X, Y)(Z)S = \Delta_R f(\tilde{X}, \tilde{Y})(TZS).$$

We denote the class of functions $h$ on $\Omega \times \Omega$ whose values on $\Omega_n \times \Omega_m$ are linear mappings $V^{n \times m} \to W^{n \times m}$ satisfying the property above (with $\Delta_R f$ replaced by $h$) as

$$T^1 = T^1(\Omega; \mathcal{W}_{nc}, \mathcal{V}_{nc}).$$

Thus for a nc function $f$, $\Delta_R f \in T^1$. 
More generally, we define the class of nc functions of order $k$,

$$\mathcal{T}^k = \mathcal{T}^k(\Omega; \mathcal{W}_{0,\text{nc}}, \mathcal{W}_{1,\text{nc}}, \ldots, \mathcal{W}_{k,\text{nc}})$$

as a class of functions on $\Omega^{k+1}$, where $\Omega \subseteq \mathcal{V}_{\text{nc}}$ is a nc set, whose values on $\Omega_{n_0} \times \cdots \times \Omega_{n_k}$ are $k$-linear forms

$$\mathcal{W}_1^{n_0 \times n_1} \times \cdots \times \mathcal{W}_k^{n_{k-1} \times n_k} \rightarrow \mathcal{W}_0^{n_0 \times n_k},$$

and which respect direct sums and similarities, or equivalently, respect intertwinings...

The class $\mathcal{T}^0 = \mathcal{T}^0(\Omega; \mathcal{W}_{\text{nc}})$ is the class of nc functions $f : \Omega \rightarrow \mathcal{W}_{\text{nc}}$. We define $\Delta_R : \mathcal{T}^k \rightarrow \mathcal{T}^{k+1}$ so that iterations $\Delta_R^\ell : \mathcal{T}^k \rightarrow \mathcal{T}^{k+\ell}$ are well defined...
Theorem

Let $f \in T^0(\Omega; \mathcal{W}_{nc})$. Then

$$\Delta^\ell_R f(X^0, \ldots, X^\ell)(Z^1, \ldots, Z^\ell) = f$$

$$= f \left( \begin{bmatrix} X^0 & Z^1 & 0 & \ldots & 0 \\ 0 & X^1 & \ldots & \ldots & \vdots \\ \vdots & \ldots & \ldots & \ldots & 0 \\ \vdots & \ldots & X^{\ell-1} & Z^\ell \\ 0 & \ldots & \ldots & 0 & X^\ell \end{bmatrix} \right)^{1,\ell+1}.$$
We use the calculus of higher order nc difference-differential operators to derive a nc analogue of the Brook Taylor expansion, which we call the \textit{Taylor–Taylor (TT) expansion in honour of Brook Taylor and of Joseph L. Taylor}.

**Theorem**

Let $f \in \mathcal{T}^0(\Omega; \mathcal{W}_{\text{nc}})$ with $\Omega \subseteq \mathcal{V}_{\text{nc}}$ a nc set, $n \in \mathbb{N}$, and $Y \in \Omega_n$. Then for each $N \in \mathbb{N}$ and arbitrary $X \in \Omega_n$,

$$f(X) = \sum_{\ell=0}^{N} \Delta_R^\ell f(Y, \ldots, Y)(X - Y, \ldots, X - Y)$$

$\ell + 1$ times $\ell$ times

$$+ \Delta_R^{N+1} f(Y, \ldots, Y, X)(X - Y, \ldots, X - Y).$$

$N+1$ times $N+1$ times
In the case where $\mathcal{V} = \mathbb{K}^d$, we obtain

\[
f(X) = \sum_{\ell=0}^{N} \sum_{w=g_{i_1} \cdots g_{i_\ell}} \Delta_{R}^{w} f(Y, \ldots, Y)(X_{i_1} - Y_{i_1}, \ldots, X_{i_\ell} - Y_{i_\ell})^{\ell+1 \text{ times}}
\]

\[
+ \sum_{w=g_{i_1} \cdots g_{i_{N+1}}} \Delta_{R}^{w} f(Y, \ldots, Y, X)(X_{i_1} - Y_{i_1}, \ldots, X_{i_{N+1}} - Y_{i_{N+1}})^{N+1 \text{ times}},
\]

where for a word $w = g_{i_1} \cdots g_{i_\ell}$,

\[
\Delta_{R}^{w} := \Delta_{R, i_\ell} \cdots \Delta_{R, i_1}.
\]

If $Y = (\mu_1 l_n, \ldots, \mu_d l_n)$, this is a genuine nc power expansion

\[
f(X) = \sum_{\ell=0}^{N} \sum_{|w|=\ell} (X - l_n\mu)^w \Delta_{R}^{w} f(\mu, \ldots, \mu)^{\ell+1 \text{ times}}
\]

\[
+ \sum_{|w|=N+1} (X - l_n\mu)^w \Delta_{R}^{w} f(\mu, \ldots, \mu, X)^{N+1 \text{ times}}.
\]
Theorem

Let $f$ be a nc function on $\mathbb{K}_\text{nc}^d$, where $\mathbb{K}$ is a field of characteristic zero, with values in $\mathcal{W}_\text{nc}$ (so that $\mathcal{W}$ is a vector space over $\mathbb{K}$). Assume that for each $n$, $f(X_1, \ldots, X_d)$ is a polynomial in $dn^2$ commuting variables $(X_i)_{jk}$, $i = 1, \ldots, d; j, k = 1, \ldots, n$, with values in $\mathcal{W}^{n \times n}$. Assume also that the degrees of these polynomials of the commuting variables $(X_i)_{jk}$ are bounded, (i.e., a degree bound is independent of $n$). Then $f$ is a nc polynomial with coefficients in $\mathcal{W}$. 
Recall that a Banach space $\mathcal{W}$ over $\mathbb{C}$ is called an operator space if a sequence of norms $\| \cdot \|_n$ on $\mathcal{W}^{n \times n}$, $n = 1, 2, \ldots$ is defined so that the following two conditions hold:

- For every $n, m \in \mathbb{N}$, $X \in \mathcal{W}^{n \times n}$ and $Y \in \mathcal{W}^{m \times m}$,
  \[
  \| X \oplus Y \|_{n+m} = \max \{ \| X \|_n, \| Y \|_m \}.
  \]

- For every $n \in \mathbb{N}$, $X \in \mathcal{W}^{n \times n}$ and $S, T \in \mathbb{C}^{n \times n}$,
  \[
  \| SXT \|_n \leq \| S \| \| X \|_n \| T \|,
  \]

where $\| \cdot \|$ denotes the operator norm of $\mathbb{C}^{n \times n}$ with respect to the standard Euclidean norm of $\mathbb{C}^n$. 

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Noncommutative functions: Algebraic and analytic results
Let $\mathcal{W}$ be an operator space. For $Y \in \mathcal{W}^{s \times s}$ and $r > 0$, define a \textit{nc ball centered at $Y$ of radius $r$} as

$$B_{nc}(Y, r) = \bigcap_{m=1}^{\infty} B \left( \bigoplus_{\alpha=1}^{m} Y, r \right)$$

$$= \bigcap_{m=1}^{\infty} \left\{ X \in \mathcal{W}^{ms \times ms} : \left\| X - \bigoplus_{\alpha=1}^{m} Y \right\|_{ms} < r \right\}.$$

\textbf{Proposition}

Let $Y \in \mathcal{W}^{s \times s}$ and $r > 0$. For any $X \in B_{nc}(Y, r)$ there is a $\rho > 0$ such that $B_{nc}(X, \rho) \subseteq B_{nc}(Y, r)$. Hence, nc balls form a basis for a topology on $\mathcal{W}_{nc}$ (the uniformly-open topology).

Open sets in this topology will be called uniformly open.
Let $V, W$ be operator spaces, and let $\Omega \subseteq V_{\text{nc}}$ be a uniformly open nc set. A nc function $f : \Omega \to W_{\text{nc}}$ is called \textit{uniformly locally bounded} if for any $s \in \mathbb{N}$ and $Y \in \Omega_s$ there exists a $r > 0$ such that $B_{\text{nc}}(Y, r) \subseteq \Omega$ and $f$ is bounded on $B_{\text{nc}}(Y, r)$, i.e., there is a $M > 0$ such that $\|f(X)\|_{sm} \leq M$ for all $m \in \mathbb{N}$ and $X \in B_{\text{nc}}(Y, r)_{sm}$.

A nc function $f : \Omega \to W_{\text{nc}}$ is called \textit{Gâteaux (G-) differentiable} if for every $n \in \mathbb{N}$ the function $f|_{\Omega_n}$ is G-differentiable, i.e., for every $X \in \Omega_n$ and $Z \in V_{n \times n}$ the G-derivative of $f$ at $X$ in direction $Z$,

$$\delta f(X)(Z) = \lim_{t \to 0} \frac{f(X + tZ) - f(X)}{t} = \frac{d}{dt} f(X + tZ) \bigg|_{t=0},$$

exists. A nc function is called \textit{uniformly analytic} if $f$ is uniformly locally bounded and G-differentiable.
Theorem

Let a nc function $f : \Omega \to \mathcal{W}_{nc}$ be uniformly locally bounded. For $s \in \mathbb{N}$, $Y \in \Omega_s$, let $\delta := \sup\{ r > 0 : f \text{ is bounded on } B_{nc}(Y, r) \}$. Then

$$f(X) = \sum_{\ell=0}^{\infty} \Delta^\ell_R f \left( \begin{array}{c} m \bigoplus_{\alpha=1} Y, \ldots, m \bigoplus_{\alpha=1} Y \\ \alpha=1 \end{array} \right) \left( \begin{array}{c} X - \bigoplus_{\alpha=1} Y, \ldots, X - \bigoplus_{\alpha=1} Y \\ \alpha=1 \end{array} \right)$$

holds, with the $TT$ series converging absolutely and uniformly, on every open nc ball $B_{nc}(Y, r)$ with $r < \delta$. 
Corollary

Let $\Omega \subseteq \mathcal{V}_{nc}$ be a uniformly open nc set. Then a nc function $f : \Omega \to \mathcal{W}_{nc}$ is uniformly locally bounded iff $f$ is continuous with respect to the uniformly-open topologies on $\mathcal{V}_{nc}$ and $\mathcal{W}_{nc}$ iff $f$ is uniformly analytic.
THANK YOU!