Matrix Inequalities and Convexity

Harry Dym	Weitzman Institute
Jeremy Greene	UC San Diego
Damon Hay	University of North Florida
Bill Helton	UC San Diego
Igor Klep	Slovenia (everywhere)
Adrian Lim	Cornell
Mihai Putinar	UC Santa Barbara
Me - Scott McCullough	U Florida
Victor Vinnikov	Ben Gurion University

Bill is 65 UCSD October 2010

NC polynomials

$\mathbb{R}\langle x\rangle$ - **polynomials in the freely non-commuting variables** $x = (x_1, \ldots, x_g)$ - *nc polys*.

The variables are *symmetric*, $x_j^T = x_j$ **and** $(pq)^T = q^T p^T$.

$p \in \mathbb{R}\langle x\rangle$ **is symmetric if** $p^T = p$.

$$\begin{aligned} q = q(x_1, x_2) =& 3 - 2x_1 + 5x_1x_2x_1 \\ r = r(x_1, x_2) =& 3 - 2x_1x_2 + 5x_2x_1 \\ r^T = r^T(x_1, x_2) =& 3 - 2x_2x_1 + 5x_1x_2. \end{aligned}$$

p **is** *symmetric* **if** $p = p^T$. **In particular,** q **is symmetric, but** r **is not.**

Evaluating NC polynomials

$\mathbb{S}^g(n)$ - g-tuples $X = (X_1, \ldots, X_g)$ of symmetric $n \times n$ matrices.

$X \in \mathbb{S}^g(n)$ corresponds to a repn $\mathbb{R}\langle x \rangle \to M_n$, $p \to p(X)$

For instance, with

$$q(x_1, x_2) = 3 - 2x_1 + 5x_1x_2x_1,$$

and $X = (X_1, X_2) \in \mathbb{S}_2(n)$,

$$q(X) = q(X_1, X_2) = 3I_n - 2X_1 + 5X_1X_2X_1.$$

If $p \in \mathbb{R}\langle x \rangle$ is symmetric, then so is the matrix $p(X)$.

Convex nc polynomials

Divide $x = (a, x)$ into two classes of variables.

A symmetric $p \in \mathbb{R}\langle a, x\rangle$ is *convex in* x (on some domain) if

$$p(A, tX + (1-t)Y) \preceq tp(A, X) + (1-t)p(A, Y).$$

The polynomial $p(x) = x^4$ ($g = 1$ and no a) is not convex. It is not too hard to find $X, Y \in M_2$ (not commuting of course) for which $(\frac{X+Y}{2})^4 \not\preceq \frac{1}{2}(X^4 + Y^4)$.

<u>Theorem</u>. If $p(a, x)$ is convex in x, then

$$p = \ell(a, x) + V(a, x)^T M(a) V(a, x),$$

where ℓ has degree at most one in x; $V(a, x)$ is linear in x; and $M(A) \succ 0$ for all A. In particular, p has degree two in x. The converse holds also.

Convex nc polys, rational functions, and LMI sets

<u>Sample Theorem</u>. If $p(a,x)$ is convex in x and concave in a, then $p = \ell(a,x) + P(x)^T P(x) - Q(a)^T Q(a)$, where P, Q are linear and ℓ has degree at most one in a and x separately.

In case there are no a variables, convexity of p near 0 implies $p = \ell(x) + P(x)^T P(x)$. In particular,

$$-p(X) \succ 0 \leftrightarrow L(X) = \begin{pmatrix} I & P(X) \\ P(X)^T & -\ell(X) \end{pmatrix} \succ 0.$$

If $-p(0) = I$, then L is a monic affine linear pencil.

In particular, $p(x) = x^4$ is not convex.

- A similar result holds for nc rational functions $r(x)$.

Matrix convex nc semialgebraic sets

Given $p \in \mathbb{R}\langle x \rangle$ symmetric, and $p(0) = I$, let

$$\mathcal{P}_p(n) = \{X \in \mathbb{S}^g(n) : p(X) \succ 0\}.$$

The sequence

$$\mathcal{P}_p = (\mathcal{P}_p(n))$$

is a *nc basic semialgebraic set***.**

The set $\mathcal{P}$ is *convex* **if each $\mathcal{P}_p(n)$ is - $\mathcal{P}_p$ is** *matrix convex***.**

Matrix convex nc semialgebraic sets

If p is concave, then $\mathcal{P}_p = \{X : p(X) \succ 0\}$ is a convex nc basic semialgebriac set.

If $p \in \mathbb{R}\langle x \rangle$ symmetric, and $r = I + X^2$, then $\mathcal{P}_p = \mathcal{P}_{rpr}$.

<u>Theorem</u>. Given a symmetric $p \in \mathbb{R}\langle x \rangle$, if $\mathcal{P}_p$ is bounded and convex and p satisfies certain irreducibility and smoothness (at the boundary of $\mathcal{P}_p$) conditions, then $-p$ is in fact convex. So convexity of one level set implies convexity of all.

The nc set $\{1 - x_1^4 - x_2^4 \succ 0\}$ is not convex.

The middle matrix and border vector

The proofs exploit the fact that convexity corresponds to some positivity (and not much is needed) of the Hessian,

$$p''(x)[h]$$

of p.

The Hessian has a representation

$$V(x)^T h M(x) h V(x).$$

Positivity of the Hessian implies $M(x)$ is some positive by the Camino-Helton-Skelton-Yi Lemma.

Because of its structure, if $M(x)$ some positive iff p has degree two (and M is constant).

Matrix-valued polynomials

Given a symmetric

$$p = \sum C_j \otimes p_j \in \mathbb{M}_\ell \otimes \mathbb{R}\langle x \rangle$$
$$p^T = \sum C_j^T \otimes p_j^T = p,$$

and $X \in \mathbb{S}^g(n)$,

$$p(X) = \sum C_j \otimes p_j(X) \in \mathbb{S}_{n\ell}.$$

As an example, given $A_j \in \mathbb{S}_d$

$$L(x) = I - \sum A_j \otimes x_j$$

is a monic affine linear pencil.

Nc convex semialgebraic sets are LMI domains

Given, $L(x) = I - \sum A_j x_j$, the nc set

$$\mathcal{P}_L = \{X : L(X) \succ 0\}$$

is convex. It is an *LMI domain.*

<u>**Theorem**</u>**. Suppose $p = \sum C_j \otimes p_j \in \mathbb{S}_\ell \otimes \mathbb{R}\langle x \rangle$ is symmetric, and $p(0) \succ 0$. If $\mathcal{P}_p$ is bounded and convex, then there is an L such that $\mathcal{P}_p = \mathcal{P}_L$; i.e., $\mathcal{P}_p$ is an LMI domain.**

• The existence of an L with `operator coefficients` **(A_j) is standard; the challenge is to get matrix coefficients.**

• There is a bound on the size of L depending only on the number of variables, ℓ, and the degree of p.

TV screen example

The nc set $\mathcal{P}_p = \{1 - x_1^4 - x_2^4 \succ 0\}$ is not convex. If it were, then it would be an LMI domain and $\mathcal{P}_p(1)$ would have an LMI representation, contradicting the real zeros condition of Bill and Victor.

Moreover, if $\mathcal{D}$ is the projection of an LMI domain $\mathcal{P}_L$; i.e.,

$$\mathcal{D} = \{X : \exists Y \text{ s.t. } L(X, Y) \succ 0\}$$

and $\mathcal{D}(1) = \mathcal{P}_L(1)$, then $\mathcal{D}$ is not a nc basic semialgebraic set since $\mathcal{D}$ is convex and $\mathcal{D}(1)$ is not LMI representable. Hence projections of LMI domains need not be basic nc semialgebraic.

What's next

- Fully incorporate a variables.
- What if all sublevel sets for p are convex?
- Projections of LMI domains, $\{X : \exists Y \text{ s.t. } L(X,Y) \succ 0\}$.
- Change variables to achieve convexity.