Matrix Inequalities and Convexity

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NC polynomials

\( \mathbb{R}\langle x \rangle \) - polynomials in the freely non-commuting variables
\( x = (x_1, \ldots, x_g) \) - nc polys.

The variables are symmetric, \( x^T_j = x_j \) and \((pq)^T = q^Tp^T\).

\( p \in \mathbb{R}\langle x \rangle \) is symmetric if \( p^T = p \).

\[
\begin{align*}
q &= q(x_1, x_2) = 3 - 2x_1 + 5x_1x_2x_1 \\
r &= r(x_1, x_2) = 3 - 2x_1x_2 + 5x_2x_1 \\
r^T &= r^T(x_1, x_2) = 3 - 2x_2x_1 + 5x_1x_2.
\end{align*}
\]

\( p \) is symmetric if \( p = p^T \). In particular, \( q \) is symmetric, but \( r \) is not.
Evaluating NC polynomials

\( S^g(n) \) - \( g \)-tuples \( X = (X_1, \ldots, X_g) \) of symmetric \( n \times n \) matrices.

\( X \in S^g(n) \) corresponds to a repn \( \mathbb{R}\langle x \rangle \rightarrow M_n, \ p \rightarrow p(X) \)

For instance, with

\( q(x_1, x_2) = 3 - 2x_1 + 5x_1x_2x_1, \)

and \( X = (X_1, X_2) \in S_2(n), \)

\( q(X) = q(X_1, X_2) = 3I_n - 2X_1 + 5X_1X_2X_1. \)

If \( p \in \mathbb{R}\langle x \rangle \) is symmetric, then so is the matrix \( p(X) \).
Convex nc polynomials

Divide \( x = (a, x) \) into two classes of variables.

A symmetric \( p \in \mathbb{R}\langle a, x \rangle \) is convex in \( x \) (on some domain) if
\[
p(A, tX + (1 - t)Y) \preceq tp(A, X) + (1 - t)p(A, Y).
\]

The polynomial \( p(x) = x^4 \) (\( g = 1 \) and no \( a \)) is not convex. It is not too hard to find \( X, Y \in M_2 \) (not commuting of course) for which \( \left( \frac{X + Y}{2} \right)^4 \not\preceq \frac{1}{2}(X^4 + Y^4) \).

**Theorem.** If \( p(a, x) \) is convex in \( x \), then
\[
p = \ell(a, x) + V(a, x)^T M(a) V(a, x),
\]
where \( \ell \) has degree at most one in \( x \); \( V(a, x) \) is linear in \( x \); and \( M(A) \succeq 0 \) for all \( A \). In particular, \( p \) has degree two in \( x \). The converse holds also.
Convex nc polys, rational functions, and LMI sets

Sample Theorem. If $p(a, x)$ is convex in $x$ and concave in $a$, then $p = \ell(a, x) + P(x)^T P(x) - Q(a)^T Q(a)$, where $P, Q$ are linear and $\ell$ has degree at most one in $a$ and $x$ separately.

In case there are no $a$ variables, convexity of $p$ near 0 implies $p = \ell(x) + P(x)^T P(x)$. In particular,

$$-p(X) \succ 0 \iff L(X) = \begin{pmatrix} I & P(X) \\ P(X)^T & -\ell(X) \end{pmatrix} \succ 0.$$  

If $-p(0) = I$, then $L$ is a monic affine linear pencil.

In particular, $p(x) = x^4$ is not convex.

- A similar result holds for nc rational functions $r(x)$. 

Matrix convex nc semialgebraic sets

Given \( p \in \mathbb{R}\langle x \rangle \) symmetric, and \( p(0) = I \), let

\[
\mathcal{P}_p(n) = \{ X \in \mathbb{S}^g(n) : p(X) \succ 0 \}.
\]

The sequence

\[
\mathcal{P}_p = (\mathcal{P}_p(n))
\]

is a nc basic semialgebraic set.

The set \( \mathcal{P} \) is convex if each \( \mathcal{P}_p(n) \) is - \( \mathcal{P}_p \) is matrix convex.
Matrix convex nc semialgebraic sets

If $p$ is concave, then $\mathcal{P}_p = \{X : p(X) \succ 0\}$ is a convex nc basic semialgebraic set.

If $p \in \mathbb{R}\langle x \rangle$ symmetric, and $r = I + X^2$, then $\mathcal{P}_p = \mathcal{P}_{rpr}$.

**Theorem.** Given a symmetric $p \in \mathbb{R}\langle x \rangle$, if $\mathcal{P}_p$ is bounded and convex and $p$ satisfies certain irreducibility and smoothness (at the boundary of $\mathcal{P}_p$) conditions, then $-p$ is in fact convex. So convexity of one level set implies convexity of all.

The nc set $\{1 - x_1^4 - x_2^4 \succ 0\}$ is not convex.
The middle matrix and border vector

The proofs exploit the fact that convexity corresponds to some positivity (and not much is needed) of the Hessian, $p''(x)[h]$ of $p$.

The Hessian has a representation

$$V(x)^T h M(x) h V(x).$$

Positivity of the Hessian implies $M(x)$ is some positive by the Camino-Helton-Skelton-Yi Lemma.

Because of its structure, if $M(x)$ some positive iff $p$ has degree two (and $M$ is constant).
Matrix-valued polynomials

Given a symmetric

\[ p = \sum C_j \otimes p_j \in \mathbb{M}_\ell \otimes \mathbb{R}\langle x \rangle \]
\[ p^T = \sum C_j^T \otimes p_j^T = p, \]

and \( X \in S^n(g) \),

\[ p(X) = \sum C_j \otimes p_j(X) \in S_{n\ell}. \]

As an example, given \( A_j \in S_d \)

\[ L(x) = I - \sum A_j \otimes x_j \]

is a monic affine linear pencil.
Nc convex semialgebraic sets are LMI domains

Given, \( L(x) = I - \sum A_j x_j \), the nc set

\[
\mathcal{P}_L = \{ X : L(X) \succ 0 \}
\]
is convex. It is an LMI domain.

**Theorem.** Suppose \( p = \sum C_j \otimes p_j \in S_\ell \otimes \mathbb{R}\langle x \rangle \) is symmetric, and \( p(0) \succ 0 \). If \( \mathcal{P}_p \) is bounded and convex, then there is an \( L \) such that \( \mathcal{P}_p = \mathcal{P}_L \); i.e., \( \mathcal{P}_p \) is an LMI domain.

- The existence of an \( L \) with operator coefficients \((A_j)\) is standard; the challenge is to get matrix coefficients.

- There is a bound on the size of \( L \) depending only on the number of variables, \( \ell \), and the degree of \( p \).
**TV screen example**

The nc set $\mathcal{P}_p = \{1 - x_1^4 - x_2^4 > 0\}$ is not convex. If it were, then it would be an LMI domain and $\mathcal{P}_p(1)$ would have an LMI representation, contradicting the real zeros condition of Bill and Victor.

Moreover, if $\mathcal{D}$ is the projection of an LMI domain $\mathcal{P}_L$; i.e.,

$$\mathcal{D} = \{X : \exists Y \text{ s.t. } L(X, Y) \succ 0\}$$

and $\mathcal{D}(1) = \mathcal{P}_L(1)$, then $\mathcal{D}$ is not a nc basic semialgebraic set since $\mathcal{D}$ is convex and $\mathcal{D}(1)$ is not LMI representable. Hence projections of LMI domains need not be basic nc semialgebraic.
What’s next

- Fully incorporate $a$ variables.
- What if all sublevel sets for $p$ are convex?
- Projections of LMI domains, $\{X : \exists Y \text{ s.t. } L(X, Y) \succ 0\}$.
- Change variables to achieve convexity.