Helton-Howe trace formula and planar shapes

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The source

Helton, J.William; Howe, Roger E. *Integral operators: traces, index, and homology.* Proc. Conf. Operator Theory, Dalhousie Univ., Halifax 1973, Lect. Notes Math. **345**, 141-209 (1973).

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The formula

$T \in \mathcal{L}(H)$ with trace-class self-commutator $[T^*, T] \in \mathcal{C}_1(H)$

For $p(z, \overline{z}), q(z, \overline{z}) \in \mathbb{C}[z, \overline{z}]$ polynomials one defines $p(T, T^*), q(T, T^*)$ using any order of T, T^* in the NC monomials.

Then

$$trace[p(T, T^*), q(T, T^*)] = u_T(\partial_z p \partial_{\overline{z}} q - \partial_z q \partial_{\overline{z}} p),$$

where

$$u_T \in \mathcal{D}'(\mathbb{C}).$$

Example

Let $S \in \mathcal{L}(\ell^2)$ be the unilateral shift. Then $u_S = \chi_{\mathbb{D}} \frac{d\text{Area}}{\pi}$, that is

$$trace[p(S,S^*),q(S,S^*)] = \int_{\mathbb{D}} (\partial_z p \partial_{\overline{z}} q - \partial_z q \partial_{\overline{z}} p) \frac{d\text{Area}}{\pi}.$$

Sufficient to verify $trace[S^{*m}S^n, S^{*r}S^s]$

from $S^*S = I$ and $Se_k = e_{k+1}$...

Helton-Howe distribution as spectral invariant

u_T is functorial in T

$\mathrm{supp}(u_T) = \sigma(T)$

$u_T(\zeta) = -\mathrm{ind}(T - \zeta)\frac{d\mathrm{Area}}{\pi}$ for $\zeta \notin \sigma_{ess}(T)$.

In general $u_T = g_T \frac{d\mathrm{Area}}{\pi}$ with $g_T \in L^1(\mathbb{C}, d\mathrm{Area})$ (proof involving more inequalities, scattering theory, singular integrals, as developed by J. Pincus and collaborators). Today g_T is known as the *principal function* of T.

The multivariate analog of u_T is more involved, less understood but it gave an elegant proof of Atiyah-Singer index formula in the context of pseudo-differential operator calculus.

Evolution

" In 1981 I have discovered cyclic cohomology and the spectral sequence relating it to Hochschild cohomology. My original motivation came from the trace formulas of Helton-Howe and Carey-Pincus for operators with trace-class commutators" from A. Connes, *Non-Commutative Geometry*, Academic Press, 1994, pp. 12.

Douglas, R.G.; Voiculescu, Dan *On the smoothness of sphere extensions* J. Oper. Theory 6, 103-111 (1981).

Shade functions

There exists a bijective correspondence between functions $g \in L^1_{\text{comp}}(\mathbb{C}, d\text{Area}),\ 0 \le g \le 1$, and irreducible linear operators $T \in \mathcal{L}(H)$ satisfying $[T^*, T] = \xi\langle\cdot, \xi\rangle$. Specifically $g = g_T$, or

$$E(w,z) = \det((T-w)(T^*-\overline{z})(T-w)^{-1}(T^*-\overline{z})^{-1}) =$$

$$= 1 - \langle (T^*-\overline{z})^{-1})\xi, (T^*-\overline{w})^{-1})\xi\rangle =$$
$$\exp(-\frac{1}{\pi}\int \frac{g(\zeta)\mathrm{dArea}(\zeta)}{(\zeta - w)(\overline{\zeta}-\overline{z})}).$$

valid over all $\mathbb{C} \times \mathbb{C}$ and separately continuous there (K. Clancey).

Exponential transform

Let

$$a_{mn} = \int z^m \overline{z}^n g(\zeta) \mathrm{dArea}(\zeta), \quad m, n \le N,$$

be given (from measurements). Then the series transform

$$\exp(-\frac{1}{\pi} \sum_{m,n=0}^{N} a_{mn} X^{m+1} Y^{n+1}) = 1 - \sum_{j,k=0}^{N} b_{jk} X^{j+1} Y^{k+1} + O(X^{N+2}, Y^{N+2})$$

has coefficients bound by the positivity conditions

$$b_{jk} = \langle T^{*(k+1)} \xi, T^{*(j+1)} \xi \rangle, \qquad [T^*, T] = \xi \langle \cdot, \xi \rangle.$$

Ramifications

Reconstruction of g via a 2D Padé approximation scheme (finite central projection of the matrix attached to T)

Gauss type cubatures for the weight g in matrix form

Regularity of free boundaries

In the case $g = \chi_{\Omega(t)}$ where $\Omega(t)$ are planar domains following the Laplacian Growth dynamics, identification of $E(z, w)$ with the Tau function of a completely integrable hierarchy

Elimination theory on compact Riemann surfaces, with $E(z, w)$ as the correct resultant and univalence criteria for analytic functions

Quadrature domains

$\det(b_{jk})_{j,k=0}^{N} = 0$ if and only if $g = \chi_\Omega$ where

$$\int_\Omega f(\zeta)\mathrm{dArea}(\zeta) = c_1 f(a_1) + \ldots + c_N f(a_n) = \pi\langle f(T)\xi, \xi\rangle = \pi\langle f(T_0)\xi, \xi\rangle$$

for all analytic functions $f \in L^1_a(\Omega)$.

In this case the reconstruction algorithm (" moments to shape") is exact at rank N.

Applications to geometric tomography (joint work with G. Golub et al).

Hele-Shaw flows

QD (and other classes of algebraic boundaries) are preserved under Hele-Shaw flows:

$\Omega_t \subset \mathbb{C}$ nested with $z = 0 \in \Omega_t$

boundary velocity $V(\zeta) = \partial_n G_{\Omega_t}(\zeta, 0)$

has sequence of conserved quantities: $\frac{d}{dt} \int_{\Omega_t} z^n dA =, n > 0$.

Generalized lemniscates, linear pencils and sums of hermitian squares

Let $Q(z, \overline{z})$ be a hermitian polynomial. The following are equivalent:

1. There exists $A \in \mathcal{L}(\mathbb{C}^n)$ with cyclic vector ξ such that

$$Q(z, \overline{z}) = \begin{vmatrix} \xi\langle \cdot, \xi \rangle & A - z \\ A^* - \overline{z} & I \end{vmatrix};$$

2. There are polynomials $Q_k(z)$ of degree $\mathrm{degQ_k} = k$, such that

$$Q(z, \overline{z}) = |Q_N(z)|^2 - |Q_{N-1}(z)|^2 - \dots - |Q_1(z)|^2 - |Q_0(z)|^2.$$

Union of disks

If the disks $D(a_j, r_j)$ are mutually disjoint, then the equation of their union is a generalized lemniscate

$$Q(z,\overline{z}) = \prod_{j=1}^{N}[|z-a_j|^2 - r_j^2] = \begin{vmatrix} \xi\langle\cdot,\xi\rangle & A-z \\ A^*-\overline{z} & I \end{vmatrix}$$

In particular the matrix $[Q(a_j,\overline{a_k})]_{j,k=0}^{N}$ is negative definite. But a little more (a four argument kernel) is needed to characterize disjointness.

Cauchy-Riemann system

Let $[T^*, T] = \xi\langle\cdot, \xi\rangle$ acting on the Hilbert space H.

For every $\varphi \in L^2(\mathbb{C}, \mathrm{dArea})$ there exists $u : \mathbb{C} \longrightarrow H$ such that the output of the system

$$\begin{cases} \frac{\partial u}{\partial z} = Tu(z) + \varphi(z)\xi \\ \\ v(z) = \langle u(z), \frac{\xi}{\|\xi\|}\rangle \end{cases}$$

satisfies

$$\|v\|_{2,\mathbb{C}} \leq \|\varphi\|_{2,\mathbb{C}}.$$

Regularity of free boundaries

Sakai's Theorem. *Let* $\Omega \subset \mathbb{C}$ *be a domain with* $\mathrm{Area}\,\partial\Omega = 0$*. If* $\int_\Omega \frac{dA(\zeta)}{\zeta - z}$ *extends analytically across* $\partial\Omega$ *from* $\mathbb{C} \setminus \overline{\Omega}$*, then* $\partial\Omega$ *is real analytic.*

New proof: Let T be associated to Ω via $g_T = \chi_\Omega$ and extend $(T^* - \overline{z})^{-1}\xi$ analytically across $\partial\Omega$ using the Schwarz function

$$S(z) = \overline{z} + \chi_\Omega(z) + \frac{1}{\pi}\int_\Omega \frac{dA(\zeta)}{\zeta - z}.$$

Then remark that $E_T(z, z) = 1 - \|(T^* - \overline{z})^{-1}\xi\|^2 = 0$ along the boundary.

Ahlfors-Beurling inequality

Let $g \in L^1_{\rm comp}(\mathbb{C}, d\text{Area}),\ 0 \le g \le \|g\|_\infty < \infty$. Then

$$|\int \frac{g(\zeta)dA(\zeta)}{\zeta - z}|^2 \le \pi\|g\|_1\|g\|_\infty.$$

Proof: Let T with $g_T = \frac{g}{\|g\|_\infty}$. Then $\int \frac{g_T(\zeta)dA(\zeta)}{\zeta - z} = \pi\langle (T^* - \overline{z})^{-1}\xi, \xi\rangle$ and $\|(T^* - \overline{z})^{-1}\xi\| \le 1$ everywhere.

Helton-Howe formula $\|\xi\|^2 = \frac{\|g\|_1}{\pi}$.

References

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